15.3: Conservative Vector Fields and Independence of Path

Example 1: Find the work done by $\mathbf{F}(x, y) = 15x^2y^2\,\mathbf{i} + 10x^3y\,\mathbf{j}$ in moving a particle from $(0,0)$ to $(3,9)$ along the following two paths:

(a) $\mathbf{r}_1(t) = \langle t, 3t \rangle$, $0 \le t \le 3$;

(b) $\mathbf{r}_2(t) = \langle t, t^2 \rangle$, $0 \le t \le 3$.

Theorem: Fundamental Theorem of Line Integrals

Suppose *R* is an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$ that contains the piecewise smooth curve *C*, given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ or $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $a \le t \le b$.

Suppose that $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ is conservative and that the component functions *M* and *N* are continuous. (Or, in $\mathbb{R}^3$, that $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is conservative and *M*, *N*, and *P* are continuous.) Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\left(x(b), y(b)\right) - f\left(x(a), y(a)\right) \quad (\text{in } \mathbb{R}^2),$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f\left(x(b), y(b), z(b)\right) - f\left(x(a), y(a), z(a)\right) \quad (\text{in } \mathbb{R}^3)$$

where *f* is a potential function of **F** (i.e., $\mathbf{F}(x, y) = \nabla f(x, y)$ or $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$).

Note: This means that the work done by a conservative vector field is *independent of path*. No matter what path is used to move a particle from Point A to Point B, the work is the same.

Example 2: Apply the Fundamental Theorem of Line Integrals to Example 1.

$\mathbf{F}(x, y) = 15x^2y^2\,\mathbf{i} + 10x^3y\,\mathbf{j}$ along paths (a) $\mathbf{r}_1(t) = \langle t, 3t \rangle$, $0 \le t \le 3$; (b) $\mathbf{r}_2(t) = \langle t, t^2 \rangle$, $0 \le t \le 3$.

Example 3: Evaluate $\int_C (6x\,dx - 4z\,dy - (4y - 20)\,dz)$ if C is a smooth curve from $(0,0,0)$ to $(3,4,0)$.

<u>Definition</u>: A region in $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be *connected* if any two points in the region can be joined by a piecewise smooth curve lying entirely with the region.

If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is the same for every piecewise smooth curve from Point A to Point B, then we say that the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent of path*.

For open regions that are connected, the path independence of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is equivalent to **F** being conservative.

<u>Theorem</u>:

If **F** is continuous on an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$, then the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is independent of path if and only if **F** is conservative.

This theorem, when combined with the Fundamental Theorem of Line Integrals, results in the following:

<u>Theorem</u>:

Suppose $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ has continuous first partial derivatives in an open connected region *R*, and suppose that *C* is any piecewise smooth curve in *R*. Then the following conditions are equivalent.

1. **F** is conservative. That is, $\mathbf{F} = \nabla f$ for some function *f*.
2. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve *C* in *R*.

<u>Note</u>: A *closed curve* is a curve in which the beginning and ending points are the same. That is, if the curve C is given by $\mathbf{r}(t)$, $a \le t \le b$, then $\mathbf{r}(a) = \mathbf{r}(b)$.

<u>Example 4:</u> Evaluate $\int_C (\sin y\, dx + x \cos y\, dy)$ if C is a smooth curve from $\left(3, \frac{\pi}{2}\right)$ to $\left(-7, \frac{\pi}{4}\right)$.

Example 5: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle 8xy - 12x^3, 4x^2 - 4y \rangle$ and C is the path from $(0,1)$ to $(2,0)$, along the first-quadrant portion of the ellipse $\dfrac{x^2}{4} + y^2 = 1$.