15.3: Conservative Vector Fields and Independence of Path

Example 1: Find the work done by $\mathbf{F}(x,y)=15x^2y^2\,\mathbf{i}+10x^3y\,\mathbf{j}$ in moving a particle from $(0,0)$ to $(3,9)$ along the following two paths:

(a) $\mathbf{r}_1(t)=\langle t,3t\rangle$, $0\le t\le 3$;

(b) $\mathbf{r}_2(t)=\langle t,t^2\rangle$, $0\le t\le 3$.

(a) $\vec{F}(x,y)=\langle 15x^2y^2, 10x^3y\rangle$

$\vec{F}(x(t),y(t)) = \langle 15t^2(3t)^2, 10(t)^3(3t)\rangle = \langle 135t^4, 30t^4\rangle$

$d\vec{r}_1 = \vec{r}_1{}'(t)\,dt = \langle 1,3\rangle\,dt$

$\text{Work} = \int_{C_1}\vec{F}\cdot d\vec{r}_1 = \int_0^3 \langle 135t^4, 30t^4\rangle\cdot\langle 1,3\rangle\,dt$

$= \int_0^3 (135t^4+90t^4)\,dt = \int_0^3 225t^4\,dt = \frac{225t^5}{5}\Big|_0^3$

$= 45t^5\Big|_0^3 = 45[(3)^5-0^5] = 45(243) = \boxed{10935}$

(b) $\vec{F}(x(t),y(t)) = \langle 15(t)^2(t^2)^2, 10(t)^3(t^2)\rangle = \langle 15t^6, 10t^5\rangle$

$d\vec{r}_2 = \vec{r}_2{}'(t)\,dt = \langle 1,2t\rangle\,dt$

$\text{Work} = \int_{C_2}\vec{F}\cdot d\vec{r}_2 = \int_0^3\langle 15t^6, 10t^5\rangle\cdot\langle 1,2t\rangle\,dt = \int_0^3 (15t^6+20t^6)\,dt$

$= \int_0^3 35t^6\,dt = \frac{35t^7}{7}\Big|_0^3 = 5t^7\Big|_0^3 = 5(3)^7 = \boxed{10935}$

Theorem: Fundamental Theorem of Line Integrals

Suppose R is an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$ that contains the piecewise smooth curve C, given by $\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}$ or $\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$, $a\le t\le b$.

Suppose that $\mathbf{F}(x,y)=M\,\mathbf{i}+N\,\mathbf{j}$ is conservative and that the component functions M and N are continuous. (Or, in $\mathbb{R}^3$, that $\mathbf{F}(x,y,z)=M\,\mathbf{i}+N\,\mathbf{j}+P\,\mathbf{k}$ is conservative and M, N, and P are continuous.) Then,

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}=f\big(x(b),y(b)\big)-f\big(x(a),y(a)\big)\quad(\text{in } \mathbb{R}^2),$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}=f\big(x(b),y(b),z(b)\big)-f\big(x(a),y(a),z(a)\big)\quad(\text{in } \mathbb{R}^3)$$

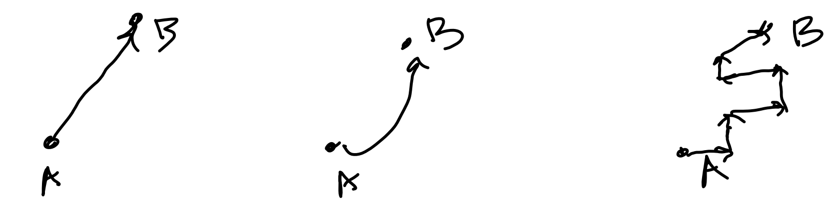
where f is a potential function of $\mathbf{F}$ (i.e., $\mathbf{F}(x,y)=\nabla f(x,y)$ or $\mathbf{F}(x,y,z)=\nabla f(x,y,z)$).

Is the vector field in Example 1 conservative?

$\vec{F}(x,y)=\langle 15x^2y^2, 10x^3y\rangle \Rightarrow \frac{\partial N}{\partial x} = 30x^2y$, $\frac{\partial M}{\partial y} = 30x^2y$

It's conservative.

Note: This means that the work done by a conservative vector field is *independent of path*. No matter what path is used to move a particle from Point A to Point B, the work is the same.



all these paths give the same $\int_C \vec{F}\cdot d\vec{r}$, if $\vec{F}$ is conservative

Example 2: Apply the Fundamental Theorem of Line Integrals to Example 1.

$\mathbf{F}(x,y) = 15x^2y^2\,\mathbf{i} + 10x^3y\,\mathbf{j}$ along paths (a) $\mathbf{r}_1(t) = \langle t, 3t\rangle$, $0 \le t \le 3$; (b) $\mathbf{r}_2(t) = \langle t, t^2\rangle$, $0 \le t \le 3$.

We already checked that $\vec{F}$ is conservative. Now find a potential function.

$f(x,y) = \int f_x\,dx = \int M\,dx = \int 15x^2y^2\,dx = 15\frac{x^3}{3}y^2 + g(y) = 5x^3y^2 + g(y)$

$f(x,y) = \int f_y\,dy = \int N\,dy = \int 10x^3y\,dy = \frac{10x^3y^2}{2} + h(x) = 5x^3y^2 + h(x)$

Let $g(y) = h(y) = K$. So $f(x,y) = 5x^3y^2 + K$. Choose $K=0$

$\text{Work} = \int_{C_1} \vec{F}\cdot d\vec{r}_1 = \int_{C_2} \vec{F}\cdot d\vec{r}_2 = f(3,9) - f(0,0) = 5x^3y^2\Big|_{(0,0)}^{(3,9)}$

$= 5(3)^3(9)^2 - 5(0)^3(0)^2 = \boxed{10935}$

Example 3: Evaluate $\int_C (6x\,dx - 4z\,dy - (4y-20)\,dz)$ if C is a smooth curve from $(0,0,0)$ to $(3,4,0)$.

Note: Equivalent problem: Find $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle 6x, -4z, -(4y-20)\rangle$, where C is a smooth curve from $(0,0,0)$ to $(3,4,0)$

Also equivalent: Find the work done by $\vec{F}(x,y,z) = \langle 6x, -4z, -(4y-20)\rangle$ on a particle moving from $(0,0,0)$ to $(3,4,0)$.

Is it conservative?

$$\operatorname{curl}\vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6x & -4z & 20-4y \end{vmatrix} = \langle -4-(-4),\ 0-0,\ 0-0\rangle$$

$= \langle 0,0,0\rangle$. So $\vec{F}$ is conservative.

So, we can evaluate $\int_C \vec{F}\cdot d\vec{r}$ by finding a potential function and applying the Fun. Thm. of Line Integrals.

Suppose $\nabla f = \vec{F}$:

$f(x,y,z) = \int f_x\,dx = \int 6x\,dx = \frac{6x^2}{2} + g(y,z) = 3x^2 + g(y,z)$

$f(x,y,z) = \int f_y\,dy = \int -4z\,dy = -4zy + h(x,z)$

$f(x,y,z) = \int f_z\,dz = \int 20-4y\,dz = 20z - 4yz + m(x,y)$

$f(x,y,z) = 3x^2 - 4zy + 20z$

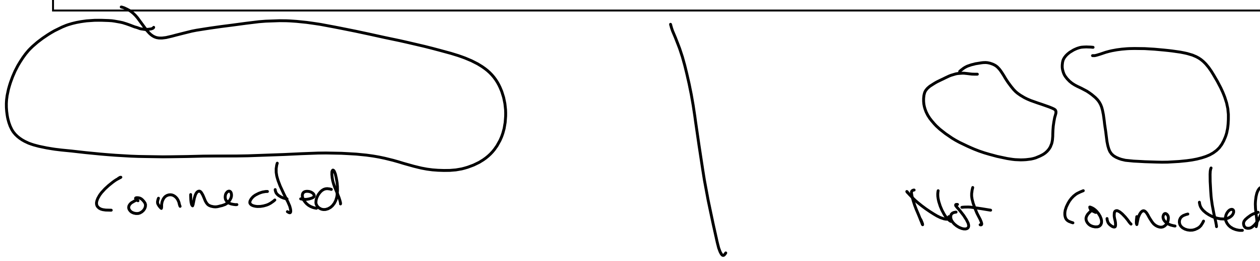
$\nabla f(x,y,z) = \langle 6x, -4z, -4y+20\rangle$ ✓

$\int_C \vec{F}\cdot d\vec{r} = f(3,4,0) - f(0,0,0)$

$= 3(3)^2 - 4(0)(4) + 20(0) - [0]$

$= \boxed{27}$

<u>Definition</u>: A region in $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be *connected* if any two points in the region can be joined by a piecewise smooth curve lying entirely with the region.



If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is the same for every piecewise smooth curve from Point A to Point B, then we say that the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent of path*.

For open regions that are connected, the path independence of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is equivalent to $\mathbf{F}$ being conservative.

<u>Theorem</u>:

If $\mathbf{F}$ is continuous on an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$, then the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

This theorem, when combined with the Fundamental Theorem of Line Integrals, results in the following:

<u>Theorem</u>:

Suppose $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ has continuous first partial derivatives in an open connected region *R*, and suppose that *C* is any piecewise smooth curve in *R*. Then the following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function *f*.
2. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve *C* in *R*.

<u>Closed curve</u>: initial point = terminal pt.

Note: A *closed curve* is a curve in which the beginning and ending points are the same. That is, if the curve C is given by $\mathbf{r}(t)$, $a \le t \le b$, then $\mathbf{r}(a) = \mathbf{r}(b)$.

Example 4: Evaluate $\int_C (\sin y\, dx + x \cos y\, dy)$ if C is a smooth curve from $\left(3, \frac{\pi}{2}\right)$ to $\left(-7, \frac{\pi}{4}\right)$.

$\vec{F}(x,y) = \langle \sin y,\ x\cos y \rangle$

$\left.\begin{aligned} \frac{\partial N}{\partial x} &= \cos y \\ \frac{\partial M}{\partial y} &= \cos y \end{aligned}\right\}$ $\vec{F}$ is conservative, so $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

Find the potential function $f(x,y)$:

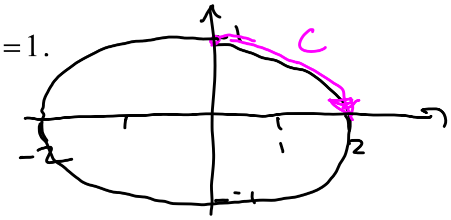
$\left.\begin{aligned} f(x,y) &= \int f_x\, dx = \int \sin y\, dx = x \sin y + g(y) \\ f(x,y) &= \int f_y\, dy = \int x \cos y\, dy = x \sin y + h(x) \end{aligned}\right\}$ $f(x,y) = x \sin y + k$

$$\int_C (\sin y\, dx + x\cos y\, dy) = f\left(-7, \tfrac{\pi}{4}\right) - f\left(3, \tfrac{\pi}{2}\right) = x \sin y \Big|_{(3, \frac{\pi}{2})}^{(-7, \frac{\pi}{4})}$$

$$= -7 \sin \frac{\pi}{4} - 3 \sin \frac{\pi}{2} = -7\left(\frac{\sqrt{2}}{2}\right) - 3(1)$$

$$= \boxed{-\frac{7\sqrt{2}}{2} - 3}$$

Example 5: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}=\langle 8xy-12x^3, 4x^2-4y\rangle$ and C is the path from $(0,1)$ to $(2,0)$, along the first-quadrant portion of the ellipse $\frac{x^2}{4}+y^2=1$.



$$\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$$

Method 1: Parametrize C:

$x=2\sin t,\ y=\cos t,\ 0\le t\le \frac{\pi}{2}$

$\vec{r}(t)=\langle 2\sin t, \cos t\rangle$

$\vec{r}\,'(t)=\langle 2\cos t, -\sin t\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$\vec{F}(x(t),y(t))$

$= \langle 8(2\sin t)(\cos t)-12(2\sin t)^3,\ 4(2\sin t)^2-4\cos t\rangle$

$= \langle 16\sin t\cos t-96\sin^3 t,\ 16\sin^2 t-4\cos t\rangle$

$$=\int_0^{\pi/2}\langle 16\sin t\cos t-96\sin^3 t, 16\sin^2 t-4\cos t\rangle\cdot\langle 2\cos t,-\sin t\rangle\,dt$$

$$=\int_0^{\pi/2}\left(32\sin t\cos^2 t-192\sin^3 t\cos t-16\sin^3 t+4\cos t\sin t\right)dt$$

Yuk! If it's conservative, we have other options.

Is it conservative? $M=8xy-12x^3$, $N=4x^2-4y$

$\frac{\partial M}{\partial y}=8x$, $\frac{\partial N}{\partial x}=8x$ Yes! It is conservative.

So, we have 2 other options:

Method 2: Find a potential function.

$$f(x,y)=\int f_x\,dx=\int M\,dx=\int(8xy-12x^3)\,dx=\frac{8x^2}{2}y-\frac{12x^4}{4}+g(y)$$
$$=4x^2y-3x^4+g(y)$$

$$f(x,y)=\int f_y\,dy=\int N\,dy=\int(4x^2-4y)\,dy=4x^2y-\frac{4y^2}{2}+h(x)$$
$$=4x^2y-2y^2+h(x)$$

Let $g(y)=-2y^2$, $h(x)=-3x^4$

$f(x,y)=4x^2y-3x^4-2y^2$

$$\int_C\vec{F}\cdot d\vec{r}=\left(4x^2y-3x^4-2y^2\right)\Big|_{(0,1)}^{(2,0)}=\left[\cancelto{0}{4(2)^2(0)}-3(2)^4-\cancelto{0}{2(0)^2}\right]-\left[\cancelto{0}{4(0)^2(1)}-\cancelto{0}{3(0)^4}-2(1)^2\right]$$
$$=-48-(-2)=-48+2=\boxed{-46}$$

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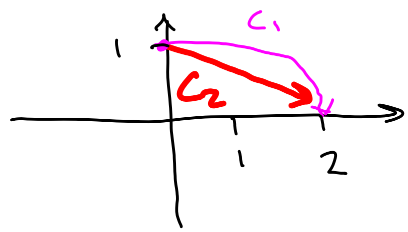
Method 3: Choose an easier path. (can only do this if $\vec{F}$ is conservative)



Parametrize C_2: $\vec{u} = \langle 2, -1 \rangle$

$\vec{r}_2(t) = \langle 0, 1 \rangle + t\langle 2, -1 \rangle$

$= \langle 0+2t, 1-t \rangle = \langle 2t, 1-t \rangle$, $0 \le t \le 1$

$\vec{F}(x,y) = \langle 8xy - 12x^3, 4x^2 - 4y \rangle$ $\qquad \vec{r}_2\,'(t) = \langle 2, -1 \rangle$

Put in $x = 2t$, $y = 1-t$:

$\vec{F}(x(t), y(t)) = \langle 8(2t)(1-t) - 12(2t)^3, 4(2t)^2 - 4(1-t) \rangle$

$= \langle 16t(1-t) - 12(8t^3), 4(4t^2) - 4 + 4t \rangle$

$= \langle 16t - 16t^2 - 96t^3, 16t^2 - 4 + 4t \rangle$

$\int_C \vec{F} \cdot d\vec{r}_2 = \int_0^1 \langle 16t - 16t^2 - 96t^3, 16t^2 - 4 + 4t \rangle \cdot \langle 2, -1 \rangle \, dt$

$= \int_0^1 (\underline{32t} - \underline{32t^2} - 192t^3 \underline{- 16t^2} + 4 - \underline{4t})\, dt$

$= \int_0^1 (-192t^3 - 48t^2 + 28t + 4)\, dt$

$= -192\frac{t^4}{4} - 48\frac{t^3}{3} + 28\frac{t^2}{2} + 4t \Big|_0^1$

$= -48t^4 - 16t^3 + 14t^2 + 4t \Big|_0^1$

$= -48 - 16 + 14 + 4 - 0 = -48 - 2 + 4 = \boxed{-46}$

Same as before

Alternate way to phrase this problem:

Evaluate $\int_C (8xy - 12x^3)\,dx + (4x^2 - 4y)\,dy$, where C is path along ellipse from $(0,1)$ to $(2,0)$.