15.4: Green's Theorem

Definition: A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, $a \le t \le b$, is said to be *simple* if $\mathbf{r}(c) \ne \mathbf{r}(d)$ for every c, d in $[a,b]$.

That is, a *simple curve* is a curve that does not intersect itself between its endpoints.

A connected region in $\mathbb{R}^2$ is said to be *simply connected* if every closed curve in R encloses only points that are in R.

That is, a *simply connected region* does not have holes. (Also, it must be connected—it cannot consist of multiple disjoint pieces.)

Green's Theorem:

Suppose R is a simply connected region in $\mathbb{R}^2$ with a piecewise smooth boundary C, oriented counterclockwise.

(That is, the region R lies to the left as C is traversed exactly once.)

Suppose $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ is a vector field with M and N having continuous first partial derivatives in an open region containing R. Then,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

Note on Notation: An integral with a circle indicates that the line integral is evaluated over a simple closed curve. Sometimes an arrow is used to indicate the orientation.

$$\oint_C M\,dx + N\,dy \quad \text{or} \quad \oint_C M\,dx + N\,dy$$

<u>Example 1:</u> Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, traversed counterclockwise.

Example 2: Evaluate $\int_C (y - x^2)\,dx + (2x - y^2)\,dy$, where C is the boundary of the region lying inside the semicircle $y = \sqrt{25 - x^2}$ and outside the semicircle $y = \sqrt{9 - x^2}$, traversed counterclockwise (positively oriented).

Example 3: Evaluate $\int_C (y + e^{\sqrt{x}})\,dx + (x^2 + \cos y^2)\,dy$, where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Example 4: Evaluate the work done by the force field $\mathbf{F}(x, y) = xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ on a particle traversing (in a counterclockwise direction) the boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

Using Green's Theorem to find area:

Sometimes the double integral over an area is easier to calculate than the line integral around the boundary. Other times, the reverse is true.

We can choose *M* and *N* strategically to come up with a formula for area:

<u>Theorem</u>: <u>Line Integral for Area</u>

Suppose *R* is a simply connected region in $\mathbb{R}^2$ with a piecewise smooth boundary *C*, oriented counterclockwise. Then the area of *R* is

$$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx\,.$$

<u>**Example 5:**</u> Find the area of the triangle with vertices $(1,2)$, $(7,3)$, and $(6,1)$.

Example 6: Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Extending Green's Theorem to a region with a hole:

Example 7: Evaluate $\int_C (y-3x^2)\,dx+(2x-\sin y)\,dy$, where C is the boundary of the region lying between the circles $x^2+y^2=4$ and $x^2+y^2=1$.