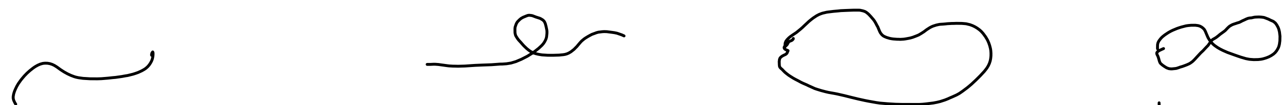
15.4: Green's Theorem

Definition: A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, $a \le t \le b$, is said to be *simple* if $\mathbf{r}(c) \ne \mathbf{r}(d)$ for every c, d in $[a, b]$.

That is, a *simple curve* is a curve that does not intersect itself between its endpoints.

A connected region in $\mathbb{R}^2$ is said to be *simply connected* if every closed curve in R encloses only points that are in R.

That is, a *simply connected region* does not have holes. (Also, it must be connected—it cannot consist of multiple disjoint pieces.)



simple, not closed

not simple, not closed

simple, closed,

not simple, closed

connected and simply connected

connected but not simply connected

not connected

Green's Theorem:

Suppose R is a simply connected region in $\mathbb{R}^2$ with a piecewise smooth boundary C, oriented counterclockwise. (positive orientation)

(That is, the region R lies to the left as C is traversed exactly once.)

Suppose $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ is a vector field with M and N having continuous first partial derivatives in an open region containing R. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

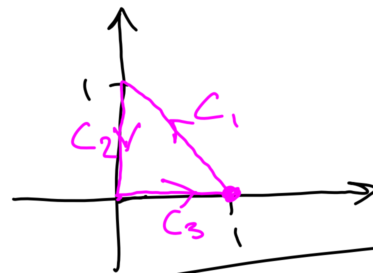
Note on Notation: An integral with a circle indicates that the line integral is evaluated over a simple closed curve. Sometimes an arrow is used to indicate the orientation.

$$\oint_C M\,dx + N\,dy \quad \text{or} \quad \oint_C M\,dx + N\,dy$$

Example 1: Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, traversed counterclockwise.

Work this problem 2 ways: ① evaluating line integral directly
② Using Green's theorem.

Method 1: Evaluate line integral directly.



Parametrize C:

$C_1: \vec r_1(t) = \langle 1,0\rangle + t\langle -1,1\rangle$
$= \langle 1-t, t\rangle, \quad 0 \le t \le 1$

$C_2: \vec r_2(t) = \langle 0,1\rangle + (t-1)\langle 0,-1\rangle$
$= \langle 0, 1-t+1\rangle = \langle 0, 2-t\rangle, \quad 1 \le t \le 2$

$C_3: \vec r_3(t) = \langle 0,0\rangle + (t-2)\langle 1,0\rangle$
$= \langle t-2, 0\rangle, \quad 2 \le t \le 3$

Note: $\vec F$ is not conservative
$M = x^4, \quad N = xy$
$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = y$

$dx_1 = -1\,dt, \quad dy_1 = 1\,dt \Leftarrow \vec r_1{}'(t) = \langle -1, 1\rangle$
$dx_2 = 0, \quad dy_2 = -1\,dt \Leftarrow \vec r_2{}'(t) = \langle 0,-1\rangle$
$dx_3 = 1\,dt, \quad dy_3 = 0 \Leftarrow \vec r_3{}'(t) = \langle 1, 0\rangle$

want $\int_C x^4\,dx + xy\,dy$

$I_1 = \int_{C_1} x^4\,dx + xy\,dy = \int_0^1 (1-t)^4(-1)\,dt + (1-t)(t)(1)\,dt$
$= -\int_0^1 (1-t)^4\,dt + \int (t-t^2)\,dt = \frac{(1-t)^5}{5}\Big|_0^1 + \frac{t^2}{2}\Big|_0^1 - \frac{t^3}{3}\Big|_0^1$
$= \frac{1}{5}(0^5 - 1^5) + \frac{1}{2} - 0 - \frac{1}{3} + 0 = -\frac{1}{5} + \frac{1}{2} - \frac{1}{3} = -\frac{6}{30} + \frac{15}{30} - \frac{10}{30} = -\frac{1}{30}$

$I_2 = \int_{C_2} x^4\,dx + xy\,dy = \int_1^2 0^4 \cancel{dx}^{\,0} + 0\,\cancel{(2-t)}^{\,0}(-1)\,dt = 0$

$I_3 = \int_{C_3} x^4\,dx + xy\,dy = \int_2^3 (t-2)^4(1)\,dt + (t-2)\cancel{(0)\,dy}^{\,0} = \int_2^3 (t-2)^4\,dt$
$= \frac{(t-2)^5}{5}\Big|_2^3 = \frac{1}{5}\left[(3-2)^5 - (2-2)^5\right] = \frac{1}{5}\left[1^5 - 0^5\right] = \frac{1}{5}$

So, integral is

$\int_C x^4\,dx + xy\,dy = I_1 + I_2 + I_3 = -\frac{1}{30} + 0 + \frac{1}{5} = -\frac{1}{30} + \frac{6}{30}$
$= \frac{5}{30} = \boxed{\frac{1}{6}}$

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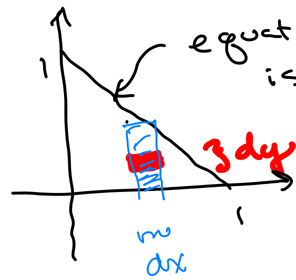
(b) Now work it using Green's Theorem.

$$\int_C x^4\,dx + xy\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$M = x^4, \quad N = xy$

$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = y$

$$= \iint_R (y-0)\,dA$$



equation of line is $y = -x+1$

$$= \int_0^1 \int_0^{-x+1} y\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2}\Big|_0^{(-x+1)} dx$$

$$= \frac{1}{2}\int_0^1 \left((-x+1)^2 - 0^2\right) dx = -\frac{1}{2}\cdot\frac{(-x+1)^3}{3}\Big|_0^1$$

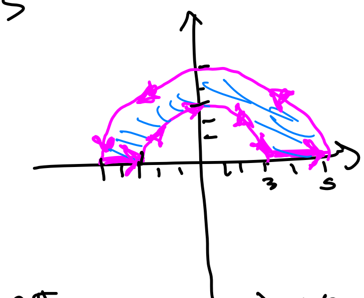
$$= -\frac{1}{6}\left[(-1+1)^3 - (0+1)^3\right] = -\frac{1}{6}\left[0^3 - 1^3\right] = -\frac{1}{6}(-1) = \boxed{\frac{1}{6}}$$

same as by evaluating line integral directly.

Example 2: Evaluate $\int_C (y-x^2)\,dx+(2x-y^2)\,dy$, where C is the boundary of the region lying inside the semicircle $y=\sqrt{25-x^2}$ and outside the semicircle $y=\sqrt{9-x^2}$, traversed counterclockwise (positively oriented).

$y^2+x^2=25$

$y^2+x^2=9$



$M=y-x^2,\quad N=2x-y^2$

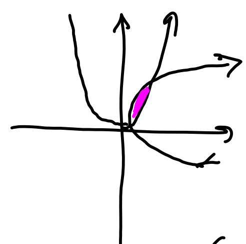
$\dfrac{\partial M}{\partial y}=1,\quad \dfrac{\partial N}{\partial x}=2$

$$\int_C (y-x^2)\,dx+(2x-y^2)\,dy=\iint_R (2-1)\,dA$$

$$=\int_0^{\pi}\int_3^5 1\,r\,dr\,d\theta=\int_0^{\pi}\frac{r^2}{2}\Big|_3^5 d\theta=\int_0^{\pi}\frac{1}{2}(5^2-3^2)\,d\theta$$

$$=\int_0^{\pi}\frac{1}{2}(16)\,d\theta=8\int_0^{\pi}d\theta=8\theta\Big|_0^{\pi}=8\pi-0=\boxed{8\pi}$$

Example 3: Evaluate $\int_C (y+e^{\sqrt{x}})\,dx+(x^2+\cos y^2)\,dy$, where C is the positively oriented boundary of the region enclosed by the parabolas $y=x^2$ and $x=y^2$.



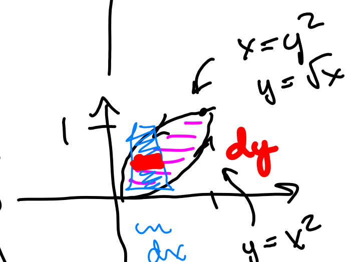
$\dfrac{\partial M}{\partial y}=1,\quad \dfrac{\partial N}{\partial x}=2x.$

$$\text{So } \int_C \vec{F}\cdot d\vec{r}=\iint_R (2x-1)\,dA=\int_0^1\int_{x^2}^{\sqrt{x}}(2x-1)\,dy\,dx$$

$$=\int_0^1 (2xy-y)\Big|_{x^2}^{\sqrt{x}}dx=\int_0^1 (2x\sqrt{x}-\sqrt{x}-2x(x^2)+x^2)\,dx$$

$$=\int_0^1 (2x^{3/2}-x^{1/2}-2x^3+x^2)\,dx=\frac{2x^{5/2}}{5/2}-\frac{x^{3/2}}{3/2}-\frac{2x^4}{4}+\frac{x^3}{3}\Big|_0^1$$

$$=\boxed{-\frac{1}{30}}$$



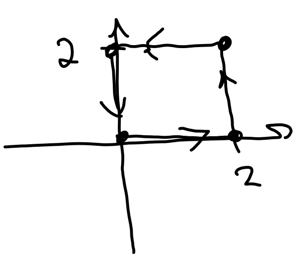
Example 4: Evaluate the work done by the force field $\mathbf{F}(x,y)=xy\mathbf{i}+(x^2+y^2)\mathbf{j}$ on a particle traversing (in a counterclockwise direction) the boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

using $dx\,dy$

$$\int_0^1\int_{y^2}^{\sqrt{y}}(2x-1)\,dx\,dy$$

$\vec{F}(x,y)=\langle xy,\ x^2+y^2\rangle$

$\dfrac{\partial M}{\partial y}=x\qquad \dfrac{\partial N}{\partial x}=2x$



$$\text{Work}=\int_C \vec{F}\cdot d\vec{r}$$

$$=\iint_R (2x-x)\,dA$$

$$=\int_0^2\int_0^2 x\,dx\,dy$$

$$=\int_0^2 \frac{x^2}{2}\Big|_0^2 dy=\int_0^2\left(\frac{2^2}{2}-\frac{0^2}{2}\right)dy$$

$$=\int_0^2 2\,dy=2y\Big|_0^2=2(2)-0=\boxed{4}$$

Using Green's Theorem to find area:

Sometimes the double integral over an area is easier to calculate than the line integral around the boundary. Other times, the reverse is true.

We can choose *M* and *N* strategically to come up with a formula for area:

To find Area, we want $\iint_R 1\,dA$. Want to find M and N such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$. Lots of ways to do it.

Let $\frac{\partial M}{\partial y} = -\frac{1}{2}$, $\frac{\partial N}{\partial x} = \frac{1}{2}$

$M = -\frac{1}{2}y$, $N = \frac{1}{2}x$

$\int_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy = \frac{1}{2}\int x\,dy - y\,dx$

$= \iint_R 1\,dA$ from Green's.

Theorem: Line Integral for Area

Suppose *R* is a simply connected region in $\mathbb{R}^2$ with a piecewise smooth boundary *C*, oriented counterclockwise. Then the area of *R* is

$$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx.$$

Alternative formulas:

Area $= \oint_C x\,dy$, Area $= -\oint_C y\,dx$

Example 5: Find the area of the triangle with vertices $(1,2)$, $(7,3)$, and $(6,1)$.

$C_1: \vec{r}_1 = \langle 1,2\rangle + t\langle 5,-1\rangle$

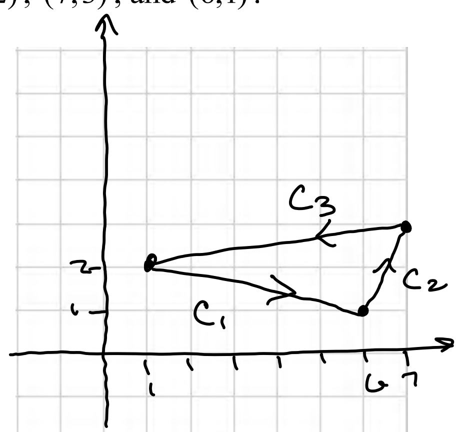
$= \langle 1+5t,\ 2-t\rangle,\ 0 \le t \le 1$

$C_2: \vec{r}_2 = \langle 6,1\rangle + (t-1)\langle 1,2\rangle,\ 1 \le t \le 2$

$= \langle 6+t-1,\ 1+2t-2\rangle = \langle 5+t,\ 2t-1\rangle$

$C_3: \vec{r}_3 = \langle 7,3\rangle + (t-2)\langle -6,-1\rangle,\ 2 \le t \le 3$

$= \langle 7-6t+12,\ 3-t+2\rangle = \langle -6t+19,\ -t+5\rangle$



$\vec{r}_1'(t) = \langle 5,-1\rangle$

$\vec{r}_2'(t) = \langle 1,2\rangle,$

$\vec{r}_3'(t) = \langle -6,-1\rangle$

Use Area $= \int_C x\,dy$

$M=0,\ N=x$

$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - 0 = 1$

So Green's tells me

$\int_C \vec{F}\cdot d\vec{r} = \iint_R 1\,dA$

$I_1: \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} x\,dy = \int_0^1 (1+5t)(-1)\,dt$

$= -\left(t + \frac{5t^2}{2}\right)\Big|_0^1 = -\left(1+\frac{5}{2}-0\right) = -\frac{2}{2} - \frac{5}{2} = -\frac{7}{2}$

$I_2: \int_{C_2} x\,dy = \int_1^2 (5+t)(2)\,dt = \int_1^2 (10+2t)\,dt = 10t + t^2\Big|_1^2 = 20+4-10-1$

$= 13$

$I_3: \int_{C_3} x\,dy = \int_2^3 (-6t+19)(-1)\,dt = \int_2^3 (6t-19) = 3t^2 - 19t\Big|_2^3 = 27-57-12+38$

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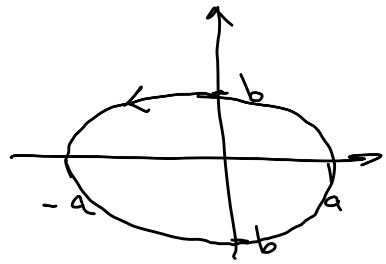
$= -30 - 12 + 38 = -42 + 38 = -4$

$$\text{Area} = I_1 + I_2 + I_3 = -\frac{7}{2} + 13 - 4 = -\frac{7}{2} + 9$$
$$= -\frac{7}{2} + \frac{18}{2} = \frac{11}{2} = \boxed{5.5}$$



Example 6: Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

$dx = -a\sin t\,dt$

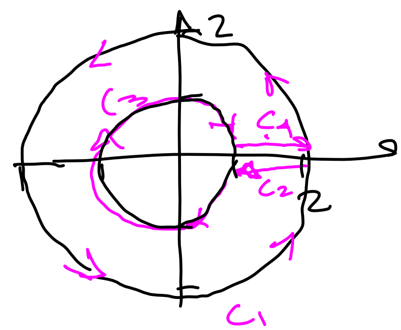
$dy = b\cos t\,dt$

$$= \frac{1}{2}\int_0^{2\pi} a\cos t\,(b\cos t)\,dt - b\sin t\,(-a\sin t\,dt)$$
$$= \frac{1}{2}\int_0^{2\pi}(ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2}ab\int_0^{2\pi}\cancelto{1}{(\cos^2 t + \sin^2 t)}\,dt$$
$$= \frac{1}{2}ab\, t\Big|_0^{2\pi} = \frac{1}{2}ab(2\pi - 0) = \boxed{\pi ab}$$

Extending Green's Theorem to a region with a hole:

(hole ⇒ not simply connected)

Example 7: Evaluate $\int_C (y-3x^2)\,dx + (2x - \sin y)\,dy$, where C is the boundary of the region lying between the circles $x^2+y^2=4$ and $x^2+y^2=1$.

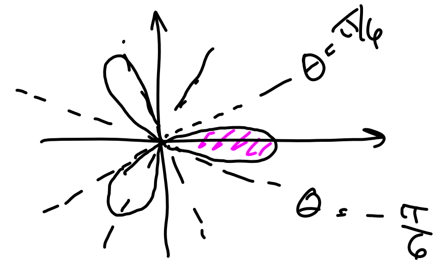


$M = y - 3x^2$, $N = 2x - \sin y$

$\frac{\partial M}{\partial y} = 1$ $\qquad \frac{\partial N}{\partial x} = 2$

$$\int_C (y-3x^2)\,dx + (2x-\sin y)\,dy$$
$$= \iint_R (2-1)\,dA = \int_0^{2\pi}\int_1^2 1\, r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_1^2\, d\theta$$
$$= \int_0^{2\pi}\left(\frac{2^2}{2} - \frac{1^2}{2}\right)d\theta = \frac{3}{2}\theta\Big|_0^{2\pi} = \frac{3}{2}(2\pi - 0) = \boxed{3\pi}$$

Example 8: Use Greens theorem to calculate the area enclosed by the polar curve $r = 3\cos 3\theta$.



Use t instead of θ:

$$r = 3\cos 3t$$

In polar, $x = r\cos t$, $y = r\sin t$

Substitute $r = 3\cos 3t$:

$x = (3\cos 3t)\cos t$, $y = (3\cos 3t)\sin t$

$x = 3\cos 3t \cos t$, $y = 3\cos 3t \sin t$

To use $\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx$, we need to find x and y in terms of t.

$$\frac{dx}{dt} = (3\cos 3t)(-\sin t) + (\cos t)(-9\sin 3t)$$
$$= -3\cos 3t \sin t - 9\cos t \sin 3t$$

$$\frac{dy}{dt} = (3\cos 3t)(\cos t) + (\sin t)(-9\sin 3t)$$
$$= 3\cos 3t \cos t - 9\sin t \sin 3t$$

$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx$. Calculate pieces separately:

these cancel out

$$x\,dy = 9\cos^2 3t\cos^2 t - \cancel{27\cos 3t\cos t\sin t\sin 3t}\quad dt$$
$$y\,dx = -9\cos^2 3t\sin^2 t - \cancel{27\cos 3t\sin t}\cos t\sin 3t\quad dt$$

Subtract:

Find $x\,dy - y\,dx = 9\cos^2 3t\cos^2 t + 9\cos^2 3t\sin^2 t$

$$= 9\cos^2 3t\,(\underbrace{\cos^2 t + \sin^2 t}_{1}) = 9\cos^2 3t\quad dt$$

$$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_{-\pi/6}^{\pi/6} 9\cos^2 3t\,dt$$

$$= \frac{1}{2}(9)\int_{-\pi/6}^{\pi/6}\frac{1}{2}\left[1 + \cos 6t\right]dt = \frac{9}{4}\left[t + \frac{1}{6}\sin 6t\right]\Bigg|_{-\pi/6}^{\pi/6}$$

$$= \frac{9}{4}\left[\frac{\pi}{6} + \frac{1}{6}\underbrace{\sin\pi}_{0} - \left(-\frac{\pi}{6}\right) - \frac{1}{6}\underbrace{\sin(-\pi)}_{0}\right] = \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \frac{9}{4}\cdot\frac{\pi}{3} = \frac{3\pi}{4}\ \text{one petal}$$

Area of whole thing: $3\left(\frac{3\pi}{4}\right) = \boxed{\frac{9\pi}{4}}$