15.5: Parametric Surfaces

A curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be represented by a vector-valued function with one parameter:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{or} \qquad \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\,.$$

A surface in $\mathbb{R}^3$ can be represented by a vector-valued function with two parameters:

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}\,).$$

Definition: Parametric Surface

Let x, y, and z be functions of u and v that are continuous on a domain D in the uv-plane. The set of points (x, y, z) given by

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

is called a *parametric surface*. The equations $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ are the parametric equations for the surface.

Example 1: Find a parametric representation for the cylinder $x^2 + y^2 = 9$.

Example 2: Find a parametric representation for the sphere $x^2 + y^2 + z^2 = a^2$.

Example 3: Write a parametric representation for the surface of revolution obtained by revolving the graph of $y = 4x^2$, $0 \le x \le 1$ around the *x*-axis. What is a parametric representation for the same graph revolved around the *y*-axis?

Tangent planes:

Suppose the parametric surface *S* is defined by $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$, and that *x*, *y*, and *z* have continuous partial derivatives in the domain *D* in the *uv*-plane. We want to find the tangent plane at the point P_0 defined by $\mathbf{r}(u_0, v_0)$.

We define the partial derivatives of **r** with respect to *u* and *v*:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u,v)\,\mathbf{i} + \frac{\partial y}{\partial u}(u,v)\,\mathbf{j} + \frac{\partial z}{\partial u}(u,v)\,\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u,v)\,\mathbf{i} + \frac{\partial y}{\partial v}(u,v)\,\mathbf{j} + \frac{\partial z}{\partial v}(u,v)\,\mathbf{k}\,.$$

If we hold $v = v_0$ constant, then $\mathbf{r}(u, v_0)$ defines a curve, in one parameter, that lies on surface *S*. Holding $u = u_0$ constant results in another curve on *S*, defined by $\mathbf{r}(u_0, v)$. The vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent to these curves (and thus to surface S) at the point P_0. The cross product, as long as it is nonzero, will therefore be a normal vector to the surface.

If $\mathbf{r}_u \times \mathbf{r}_v(u,v) \ne \mathbf{0}$ for every (u,v) in *D*, then the surface S is called *smooth* on *D*.

If *S* is smooth, a normal vector at the point $(x_0, y_0, z_0) = (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$ is given by

$$\mathbf{N} = \mathbf{r}_u(u_0,v_0) \times \mathbf{r}_v(u_0,v_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

<u>Example 4:</u> Find the equation of the tangent plane to the surface defined by $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$ at the point where $u = 1$, $v = \dfrac{\pi}{4}$.

Area of a parametric surface:

To construct an integral for the surface area, we transform rectangles in the *uv*-plane to parallelograms that are tangent to the parametric surface in $\mathbb{R}^3$.

<u>Recall</u>: Area of a parallelogram bounded by vectors **u** and **v** is:

Area of parallelogram = $\|\mathbf{u}\times\mathbf{v}\|$.

<u>Formula for Surface Area of a Parametric Surface</u>:

Let *S* be a smooth parametric surface defined by $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ in an open region *D* in the *uv*-plane. If each point on the surface *S* corresponds to exactly one point in the domain *D*, then the surface area of *S* is given by

$$\text{Surface area} = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA,$$

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}(u,v)\,\mathbf{i} + \dfrac{\partial y}{\partial u}(u,v)\,\mathbf{j} + \dfrac{\partial z}{\partial u}(u,v)\,\mathbf{k}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}(u,v)\,\mathbf{i} + \dfrac{\partial y}{\partial v}(u,v)\,\mathbf{j} + \dfrac{\partial z}{\partial v}(u,v)\,\mathbf{k}$.

Example 5: Find the area of the surface defined by $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

Example 6: Find the area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$, and $(2,1)$. (Same problem as Section 14.5 Example 1.)

Example 7: Find the area of the surface defined by $\mathbf{r}(u,v) = uv\,\mathbf{i} + (u+v)\,\mathbf{j} + (u-v)\,\mathbf{k}$, $u^2 + v^2 \le 1$.

Example 8: Find the area of the surface obtained by revolving the curve $y = \sqrt[3]{x}$, $1 \leq y \leq 2$ about the y-axis.