## 15.5: Parametric Surfaces

A curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be represented by a vector-valued function with one parameter:

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad \text{or} \qquad \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}\,.$$

A surface in $\mathbb{R}^3$ can be represented by a vector-valued function with two parameters:

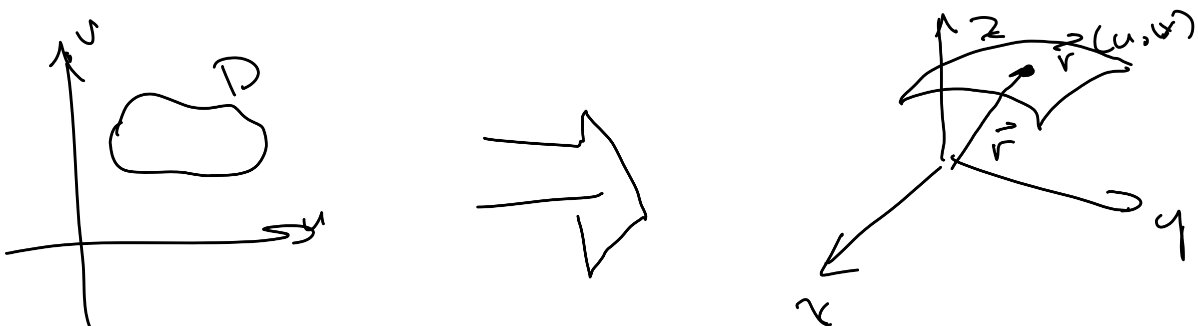
$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}\,).$$

Definition: Parametric Surface

Let $x$, $y$, and $z$ be functions of $u$ and $v$ that are continuous on a domain $D$ in the $uv$-plane. The set of points $(x, y, z)$ given by

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$$

is called a *parametric surface*. The equations $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ are the parametric equations for the surface.




**Example 1:** Find a parametric representation for the cylinder $x^2 + y^2 = 9$.

Choose: Let $z = u$

Let $v$ play the role of $\theta$

We know $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

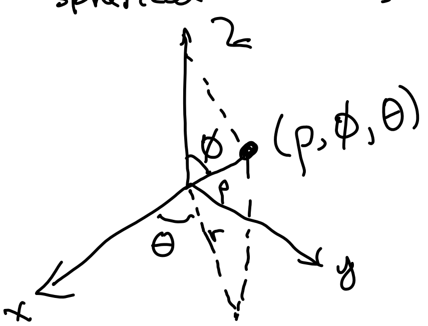
$(3\cos\theta)^2 + (3\sin\theta)^2 = 9$

Let $x = 3\cos v$, $3\sin v$

$$\boxed{\vec{r}(u,v) = \langle 3\cos v, 3\sin v, u\rangle}$$

**Example 2:** Find a parametric representation for the sphere $x^2 + y^2 + z^2 = a^2$.

spherical coordinates:




$x = r\cos\theta$, $y = r\sin\theta$, $z = \rho\cos\phi$

$r = \rho\sin\phi$

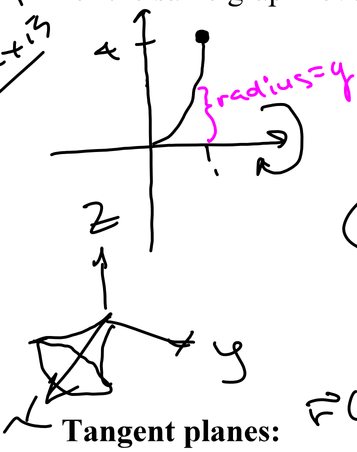
$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$. Here, $\rho = 3$

Let $u = \phi$, $v = \theta \implies x = 3\sin u\cos v$, $y = 3\sin u\sin v$

$$\boxed{\vec{r}(u,v) = \langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u\rangle,\quad 0 \le u \le \pi,\quad 0 \le v \le 2\pi}$$

**Example 3:** Write a parametric representation for the surface of revolution obtained by revolving the graph of $y = 4x^2$, $0 \le x \le 1$ around the $x$-axis. What is a parametric representation for the same graph revolved around the $y$-axis?

Around x-axis




Let $x = u$

$$\begin{cases} y = r\sin\theta \\ z = r\cos\theta \end{cases}$$

$y = 4u^2 \sin v$

$z = 4u^2 \cos v$

$\vec{r}(u,v) = \langle u, 4u^2 \sin v, 4u^2 \cos v \rangle$

See last 2 pages of notes

**Tangent planes:**

Suppose the parametric surface $S$ is defined by $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$, and that $x$, $y$, and $z$ have continuous partial derivatives in the domain $D$ in the $uv$-plane. We want to find the tangent plane at the point $P_0$ defined by $\mathbf{r}(u_0, v_0)$.

We define the partial derivatives of $\mathbf{r}$ with respect to $u$ and $v$:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u,v)\,\mathbf{i} + \frac{\partial y}{\partial u}(u,v)\,\mathbf{j} + \frac{\partial z}{\partial u}(u,v)\,\mathbf{k}$$

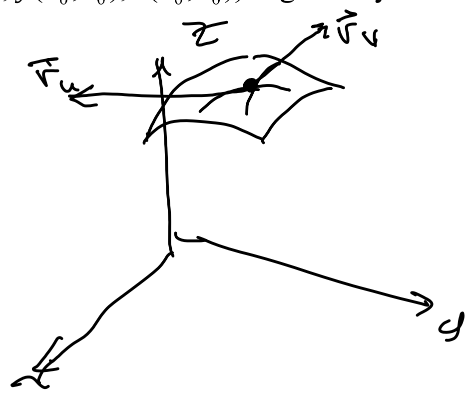
$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u,v)\,\mathbf{i} + \frac{\partial y}{\partial v}(u,v)\,\mathbf{j} + \frac{\partial z}{\partial v}(u,v)\,\mathbf{k}.$$

If we hold $v = v_0$ constant, then $\mathbf{r}(u, v_0)$ defines a curve, in one parameter, that lies on surface $S$. Holding $u = u_0$ constant results in another curve on $S$, defined by $\mathbf{r}(u_0, v)$. The vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are tangent to these curves (and thus to surface S) at the point $P_0$. The cross product, as long as it is nonzero, will therefore be a normal vector to the surface.

If $\mathbf{r}_u \times \mathbf{r}_v(u,v) \ne \mathbf{0}$ for every $(u,v)$ in $D$, then the surface S is called *smooth* on $D$.

If $S$ is smooth, a normal vector at the point $(x_0, y_0, z_0) = (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$ is given by

$$\mathbf{N} = \mathbf{r}_u(u_0,v_0) \times \mathbf{r}_v(u_0,v_0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$



**Example 4:** Find the equation of the tangent plane to the surface defined by $\mathbf{r}(u,v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$ at the point where $u = 1$, $v = \dfrac{\pi}{4}$.

$$\vec{r}(u,v) = \langle u^2, 2u\sin v, u\cos v\rangle$$

$$\vec{r}_u(u,v) = \langle 2u, 2\sin v, \cos v\rangle$$

$$\vec{r}_v(u,v) = \langle 0, 2u\cos v, -u\sin v\rangle$$

$$(\vec{r}_u \times \vec{r}_v)(u,v) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 2\sin v & \cos v \\ 0 & 2u\cos v & -u\sin v \end{vmatrix} = \langle -2u\sin^2 v - 2u\cos^2 v,\ 2u^2\sin v,\ 4u^2\cos v\rangle$$

$$= \langle -2u,\ 2u^2\sin v,\ 4u^2\cos v\rangle$$

$$(\vec{r}_u \times \vec{r}_v)\left(1, \tfrac{\pi}{4}\right) = \left\langle -2(1),\ 2(1)^2\sin\tfrac{\pi}{4},\ 4(1)^2\cos\tfrac{\pi}{4}\right\rangle = \left\langle -2, \tfrac{2\sqrt{2}}{2}, \tfrac{4\sqrt{2}}{2}\right\rangle = \langle -2, \sqrt{2}, 2\sqrt{2}\rangle$$

Point on plane: $\vec{r}\left(1, \tfrac{\pi}{4}\right) = \left\langle 1^2,\ 2(1)\sin\tfrac{\pi}{4},\ 1\cos\tfrac{\pi}{4}\right\rangle = \left\langle 1, 2\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right\rangle = \left\langle 1, \sqrt{2}, \tfrac{\sqrt{2}}{2}\right\rangle$

Write eqn of plane:

$$\boxed{-2(x-1) + \sqrt{2}(y-\sqrt{2}) + 2\sqrt{2}\left(z - \tfrac{\sqrt{2}}{2}\right) = 0}$$

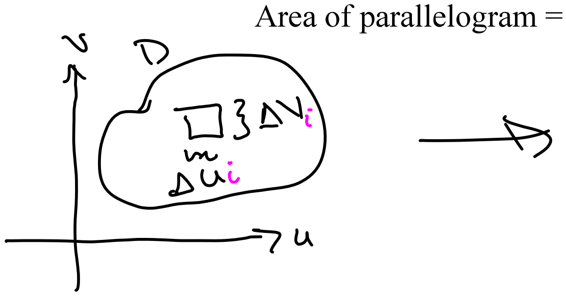
simplify it

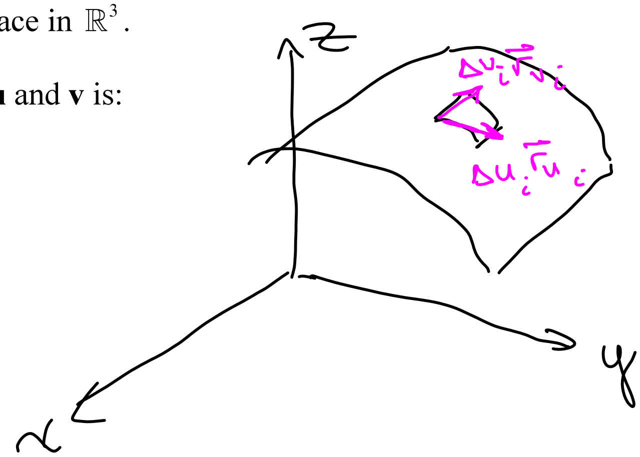
**Area of a parametric surface:**

To construct an integral for the surface area, we transform rectangles in the $uv$-plane to parallelograms that are tangent to the parametric surface in $\mathbb{R}^3$.

Recall: Area of a parallelogram bounded by vectors $\mathbf{u}$ and $\mathbf{v}$ is:

$$\text{Area of parallelogram} = \|\mathbf{u}\times\mathbf{v}\|.$$




Formula for Surface Area of a Parametric Surface:

Let $S$ be a smooth parametric surface defined by $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ in an open region $D$ in the $uv$-plane. If each point on the surface $S$ corresponds to exactly one point in the domain $D$, then the surface area of $S$ is given by

$$\text{Surface area} = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,dA,$$

dA in uv-plane

where $\mathbf{r}_u = \dfrac{\partial x}{\partial u}(u,v)\,\mathbf{i} + \dfrac{\partial y}{\partial u}(u,v)\,\mathbf{j} + \dfrac{\partial z}{\partial u}(u,v)\,\mathbf{k}$ and $\mathbf{r}_v = \dfrac{\partial x}{\partial v}(u,v)\,\mathbf{i} + \dfrac{\partial y}{\partial v}(u,v)\,\mathbf{j} + \dfrac{\partial z}{\partial v}(u,v)\,\mathbf{k}$.

Set up the integral

**Example 5:** Find the area of the surface defined by $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle$$

$$\vec{r}_u(u,v) = \langle \cos v,\ \sin v,\ 0\rangle$$

$$\vec{r}_v(u,v) = \langle -u\sin v,\ u\cos v,\ 1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v,\ -\cos v,\ u\cos^2 v + u\sin^2 v\rangle$$

$$= \langle \sin v,\ -\cos v,\ u\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$S = \iint_S \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^{\pi}\int_0^1 \sqrt{1+u^2}\,du\,dv$$

**Example 6:** Find the area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$, and $(2,1)$. (Same problem as Section 14.5 Example 1.)

Parametrize the surface: Let $x = u$, $y = v$, $z = 1 + 3u + 2v^2$

$$\vec{r}(u,v) = \langle u,\ v,\ 1+3u+2v^2\rangle$$

$$\vec{r}_u(u,v) = \langle 1, 0, 3\rangle$$

$$\vec{r}_v(u,v) = \langle 0, 1, 4v\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ 0 & 1 & 4v \end{vmatrix} \right\|$$

$$= \|\langle -3, -4v, 1\rangle\|$$

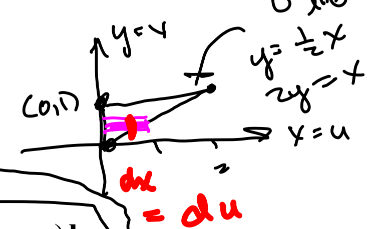
$$= \sqrt{(-3)^2 + (-4v)^2 + 1^2} = \sqrt{16v^2 + 10}$$

$$\text{Surface Area} = \iint_D \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^1\int_0^{2v} \sqrt{16v^2+10}\,du\,dv$$

$$= \int_0^1 u\sqrt{16v^2+10}\Big|_0^{2v}\,dv$$

$$= \int_0^1 2v\,(16v^2+10)^{1/2}\,dv$$

$$= 2\cdot\frac{1}{32}\cdot\frac{(16v^2+10)^{3/2}}{3/2}\Big|_0^1$$




$$2\cdot\frac{1}{32}\cdot\frac{2}{3}\left[(16(1)^2+10)^{3/2} - (16(0)^2+10)^{3/2}\right]$$

$$\frac{1}{16}\cdot\frac{2}{3}\left[26^{3/2} - 10^{3/2}\right]$$

$$\boxed{\frac{1}{24}\left[26\sqrt{26} - 10\sqrt{10}\right]}$$

**Example 7:** Find the area of the surface defined by $\mathbf{r}(u,v) = uv\,\mathbf{i} + (u+v)\,\mathbf{j} + (u-v)\,\mathbf{k}$, $u^2 + v^2 \le 1$.

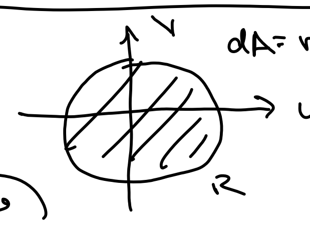
$$\vec{r}(u,v) = \langle uv,\ u+v,\ u-v\rangle$$

$$\vec{r}_u(u,v) = \langle v, 1, 1\rangle, \quad \vec{r}_v(u,v) = \langle u, 1, -1\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} \right\| = \|\langle -2,\ u+v,\ v-u\rangle\|$$

$$= \sqrt{4 + (u+v)^2 + (v-u)^2} = \sqrt{4 + u^2 + \cancel{2uv} + v^2 + v^2 - \cancel{2uv} + u^2}$$

$$= \sqrt{4 + 2u^2 + 2v^2} = \sqrt{2(2+u^2+v^2)} = \sqrt{2}\,\sqrt{2+u^2+v^2}$$




$dA = r\,dr\,d\theta$

$$S = \iint_R \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^{2\pi}\int_0^1 \sqrt{2}\,\sqrt{2+u^2+v^2}\;r\,dr\,d\theta$$

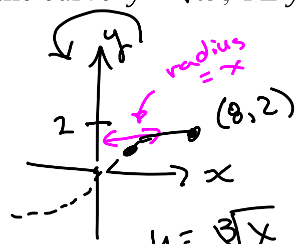
$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \sqrt{2+r^2}\,(r)\,dr\,d\theta$$

next page

$$= \sqrt{2}\cdot\frac{1}{2}\int_0^{2\pi} \frac{(2+r^2)^{3/2}}{3/2}\Big|_0^1 d\theta = \frac{\sqrt{2}}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\int_0^{2\pi}\left[(2+1^2)^{3/2}-(2+0^2)^{3/2}\right] d\theta$$

$$= \frac{\sqrt{2}}{3}\left(3^{3/2}-2^{3/2}\right)\theta\Big|_0^{2\pi} = \frac{\sqrt{2}}{3}(3\sqrt{3}-2\sqrt{2})(2\pi-0) = \left(\frac{3\sqrt{6}}{3}-\frac{4}{3}\right)2\pi = \boxed{2\pi\sqrt{6}-\frac{8\pi}{3}}$$

**Example 8:** Find the area of the surface obtained by revolving the curve $y=\sqrt[3]{x}$, $1 \le y \le 2$ about the $y$-axis.



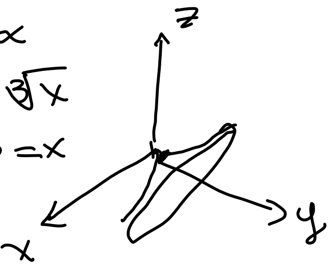

Parametrize: Let $y = u$

$$\left.\begin{aligned} x &= r\cos\theta \\ z &= r\sin\theta \end{aligned}\right\} \quad \begin{aligned} x &= u^3\cos v \\ z &= u^3 \sin v \end{aligned}$$

$r = y^3$

$y = \sqrt[3]{x}$

$y^3 = x$



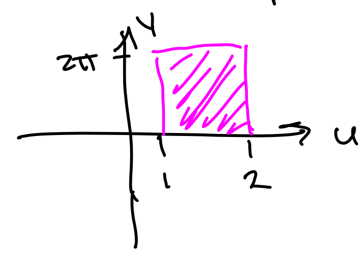

$F(u,v) = \langle u^3\cos v,\ u,\ u^3\sin v\rangle$

If you work it out, you should get $\|\vec{r}_u \times \vec{r}_v\| = \sqrt{u^6+9u^{10}} = u^3\sqrt{1+9u^4}$

Note $u \ge 0$ here

$$S = \iint_R \|\vec{r}_u \times \vec{r}_v\|\, dA$$

$dA$ is in $uv$ plane




$$= \int_0^{2\pi}\int_1^2 u^3\sqrt{1+9u^4}\,du\,dv$$

We stopped here in class and just wrote the answer.
Following are the details of the integration:

$$\int_0^{2\pi}\int_{1=u}^{2=u} \frac{1}{36}t^{1/2}\,dt\,dv$$

$t = 1+9u^4$

$\frac{dt}{du} = 36u^3$

$dt = 36u^3\,du$

$\frac{1}{36}dt = u^3\,du$

$$= \frac{1}{36}\int_0^{2\pi} \frac{t^{3/2}}{3/2}\Big|_{u=1}^{u=2} dv = \frac{1}{36}\int_0^{2\pi}\frac{2}{3}(1+9u^4)^{3/2}\Big|_{u=1}^{u=2} dv$$

$$= \frac{1}{54}\int_0^{2\pi}\left[(1+9(2)^4)^{3/2}-(1+9(1)^4)^{3/2}\right]dv = \frac{1}{54}\left[145^{3/2}-10^{3/2}\right]v\Big|_0^{2\pi}$$

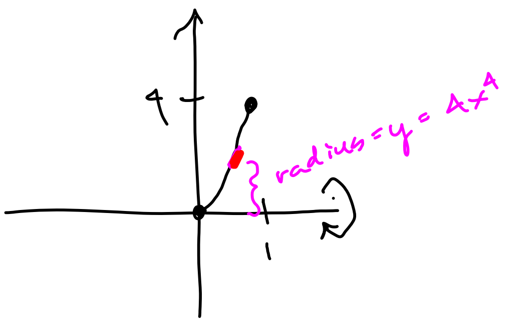
$$= \frac{1}{54}\left[145^{3/2}-10^{3/2}\right](2\pi-0) = \boxed{\frac{\pi}{27}\left[145^{3/2}-10^{3/2}\right]}$$

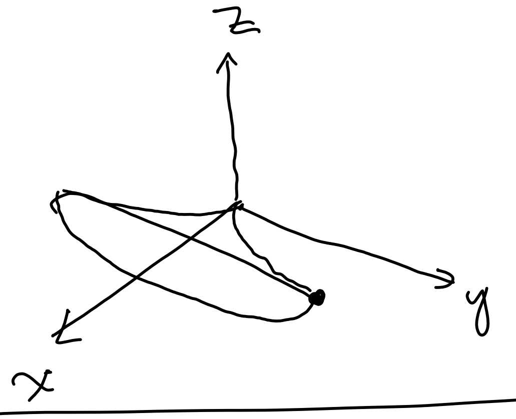
# How to parametrize a surface of rotation

**Ex 1 revisited:**

Rotate the graph of $y = 4x^4$, $0 \le x \le 1$ around ⓐ the $x$-axis, ⓑ the $y$-axis. Write parametric representation for each.

ⓐ



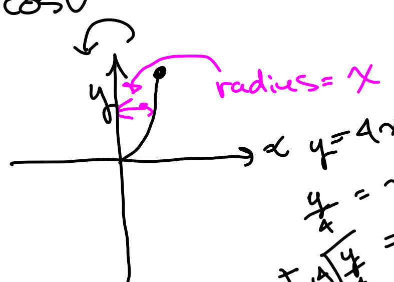




Let $x = u$

$$\left.\begin{array}{l} y = r\sin\theta \\ z = r\cos\theta \end{array}\right\} \quad \begin{array}{l} y = 4u^4 \sin v \\ z = 4u^4 \cos v \end{array}$$

$$\boxed{\vec{r}(u,v) = \langle u,\ 4u^4 \sin v,\ 4u^4 \cos v \rangle}$$

ⓑ

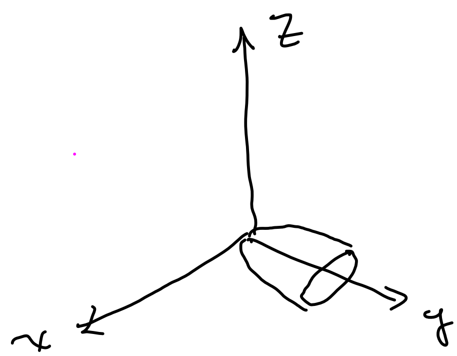



$y = 4x^4$

$\frac{y}{4} = x^4$

$\pm \sqrt[4]{\frac{y}{4}} = x$

want $(+)$

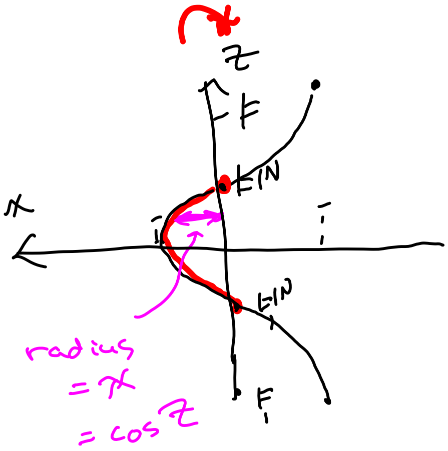


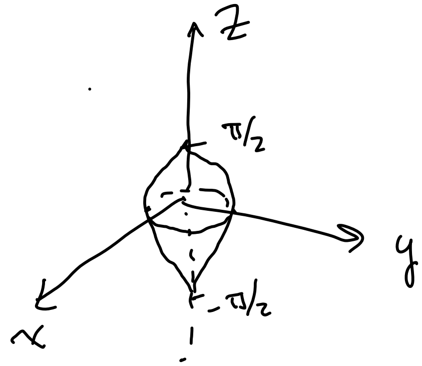

Let $y = u$

Then $\left.\begin{array}{l} x = r\cos\theta \\ z = r\sin\theta \\ \text{Use } r = \sqrt[4]{\frac{y}{4}} \end{array}\right\} \Rightarrow \begin{array}{l} x = \sqrt[4]{\frac{u}{4}}\cos v \\ z = \sqrt[4]{\frac{u}{4}}\sin v \end{array}$

$$\boxed{\vec{r}(u,v) = \left\langle \sqrt[4]{\frac{u}{4}}\cos v,\ u,\ \sqrt[4]{\frac{u}{4}}\sin v \right\rangle}$$

**Ex 1.5:** Rotate graph of $x = \cos z$, $-\frac{\pi}{2} \le z \le \frac{\pi}{2}$ around $z$-axis




$$\left.\begin{aligned} \text{Let } z &= u \\ y &= r\sin\theta \\ x &= r\cos\theta \end{aligned}\right\} \quad \begin{aligned} z &= u \\ y &= (\cos z)\sin\theta = \cos u \sin v \\ x &= (\cos z)\cos\theta = \cos u \cos v \end{aligned}$$

$$\boxed{\vec{r}(u,v) = \langle \cos u \cos v,\ \cos u \sin v,\ u \rangle}$$