15.6: Surface Integrals

Whether a surface is described by a function $z = g(x, y)$ or by a vector-valued function $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$ in two parameters, we can have a scalar-valued function $f(x, y, z)$, that produces a scalar for each point (x, y, z) on the surface.

The *surface integral* of a function *f* over a surface *S* is defined as

$$\iint_S f(x,y,z)\,dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta S_i\,,$$

where each ΔS_i is a "patch" on the surface, (x_i, y_i, z_i) is a point on ΔS_i, and that all the incremental surface patches ΔS_i partition the surface in such a way that the area of the largest patch approaches 0 as the number of patches approaches infinity. (The integral is defined only if the limit exists.)

As with a double integral over an area, or a triple integral over a volume, we evaluate a surface integral by writing it as an iterated integral.

> Theorem: Let *S* be a surface with equation $z = g(x, y)$, and let *R* be its projection onto the *xy*-plane. If *g* and its first partial derivatives are continuous on *R*, and if *f* is continuous on *S*, then the surface integral of *f* over *S* is
>
> $$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{1+[g_x(x,y)]^2+[g_y(x,y)]^2}\,dA.$$

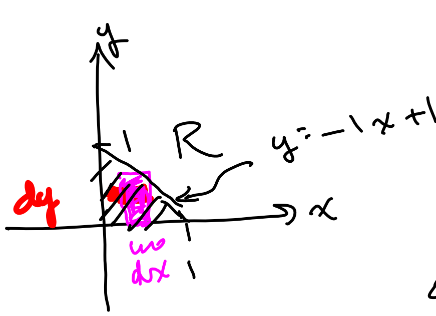
Example 1: Evaluate the surface integral $\iint_S (x+y+z)\,dS$, where *S* is the first-octant portion of the plane $2x + 2y + z = 2$.

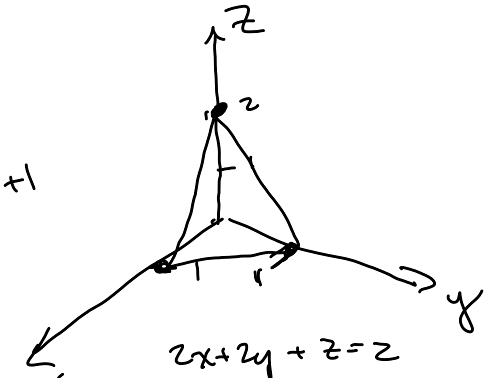
$g(x,y) = z = 2-2x-2y$

$g_x(x,y) = \dfrac{\partial z}{\partial x} = -2$

$g_y(x,y) = \dfrac{\partial z}{\partial y} = -2$

$$\iint_S (x+y+z)\,dS = \iint_R (x+y+z)\sqrt{1+(g_x)^2+(g_y)^2}\,dA$$





$2x+2y+z=2$

$x=0,\ y=0 \Rightarrow z=2$

$x=0,\ z=0 \Rightarrow 2y=2 \Rightarrow y=1$

$y=0,\ z=0 \Rightarrow 2x=2 \Rightarrow x=1$

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$$\iint_S (x+y+z)\,dS = \iint_R (x+y+z)\sqrt{1+(g_x)^2+(g_y)^2}\;dA$$

$$= \int_0^1\int_0^{-x+1} (x+y+z)\sqrt{1+(-2)^2+(-2)^2}\;dy\,dx$$

$$= \int_0^1\int_0^{-x+1} (x+y+(2-2x-2y))\sqrt{9}\;dy\,dx = 3\int_0^1\int_0^{-x+1} (-x-y+2)\,dy\,dx$$

turns out to be $\boxed{2}$

Note: $2x+2y+z=2$

Could write $G(x,y,z)=2x+2y+z-2=0$

$\nabla G(x,y,z)=\langle 2,2,1\rangle$

$dS=\|\nabla G(x,y,z)\|\,dA=\sqrt{2^2+2^2+1}\;dA=3\,dA$

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,z)\,\|\nabla G\|\,dA$$

Surface integrals over surfaces defined parametrically:

If a surface S is defined parametrically by $\mathbf{r}(u,v)=x(u,v)\,\mathbf{i}+y(u,v)\,\mathbf{j}+z(u,v)\,\mathbf{k}$, then we already know that $\iint_S dS=\iint_D\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA$. So, not surprisingly, when we calculate $\iint_S f(x,y,z)\,dS$ for parametrically defined surfaces, we simply replace dS by $\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA$.

Theorem: Surface Integral over a Parametric Surface

Suppose a surface S is described by the vector-valued function $\mathbf{r}(u,v)=x(u,v)\,\mathbf{i}+y(u,v)\,\mathbf{j}+z(u,v)\,\mathbf{k}$, on a domain D in the uv-plane. Then the surface integral of $f(x,y,z)$ over S is

$$\iint_S f(x,y,z)\,dS=\iint_D f(x(u,v),y(u,v),z(u,v))\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA=\iint_D f(\mathbf{r}(u,v))\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA.$$

Note: Compare this to the line integral for a parametrically defined curve:

$ds=\|\vec{r}\,'(t)\|\,dt$

$$\int_C f(x,y,z)\,ds=\int_a^b f(x(t),y(t),z(t))\,\|\mathbf{r}'(t)\|\,dt=\int_a^b f(\mathbf{r}'(t))\,\|\mathbf{r}'(t)\|\,dt$$

$dS=\|\nabla G(x,y,z)\|\,dA$

$dS=\|\vec{r}_u\times\vec{r}_v\|\,dA$

Example 2: Evaluate the surface integral $\iint_S yz\,dS$, where S is defined by

$\mathbf{r}(u,v) = u^2\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \frac{\pi}{2}$.

$$\vec{r}(u,v) = \langle u^2, u\sin v, u\cos v\rangle$$
$$\vec{r}_u(u,v) = \langle 2u, \sin v, \cos v\rangle$$
$$\vec{r}_v(u,v) = \langle 0, u\cos v, -u\sin v\rangle$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & \sin v & \cos v \\ 0 & u\cos v & -u\sin v \end{vmatrix}$$

$$= \langle -u\sin^2 v - u\cos^2 v,\ 0 + 2u^2\sin v,\ 2u^2\cos v - 0\rangle$$

$$= \langle -u, 2u^2\sin v, 2u^2\cos v\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{u^2 + (2u^2\sin v)^2 + (2u^2\cos v)^2} = \sqrt{u^2 + 4u^4\sin^2 v + 4u^4\cos^2 v}$$

$$= \sqrt{u^2 + 4u^4} = \sqrt{u^2}\sqrt{1+4u^2} = u\sqrt{1+4u^2}$$ (Note: $u \ge 0$, so $\sqrt{u^2} = u$)

$$\iint_S yz\,dS = \iint_R (u\sin v)(u\cos v)\|\vec{r}_u \times \vec{r}_v\|\,dA$$

$$= \int_0^{\pi/2}\int_0^1 (u\sin v)(u\cos v)\left(u\sqrt{1+4u^2}\right)du\,dv = \int_0^{\pi/2}\int_0^1 u^3 \sin v\cos v\sqrt{1+4u^2}\,du\,dv$$

$$= \int_0^{\pi/2} \sin v\cos v \underbrace{\int_0^1 u^3\sqrt{1+4u^2}\,du}_{I_1}\,dv$$

Let $I_1 = \int_0^1 u^3(1+4u^2)^{1/2}du = \int_0^1 \underbrace{u^2}_{\frac{1}{4}(t-1)} \cdot \underbrace{u\,\underbrace{(1+4u^2)^{1/2}}_{t^{1/2}}\,du}_{\frac{1}{8}dt}$

$t = 1+4u^2$
$\frac{dt}{du} = 8u$
$dt = 8u\,du$
$\frac{1}{8}dt = u\,du$
$t - 1 = 4u^2$
$\frac{1}{4}(t-1) = u^2$

$$= \frac{1}{4}\cdot\frac{1}{8}\int_{0=u}^{1=u} (t-1)t^{1/2}\,dt$$

$$= \frac{1}{32}\int_{0=u}^{1=u}\left[t^{3/2} - t^{1/2}\right]dt = \frac{1}{32}\left[\frac{t^{5/2}}{5/2} - \frac{t^{3/2}}{3/2}\right]_{u=0}^{u=1}$$

$$= \frac{1}{32}\left[\frac{2}{5}(1+4u^2)^{5/2}\Big|_0^1 - \frac{2}{3}(1+4u^2)^{3/2}\Big|_0^1\right]$$

$$= \frac{1}{32}\left[\frac{2}{5}(5^{5/2} - 1^{5/2}) - \frac{2}{3}(5^{3/2} - 1^{3/2})\right] = \frac{1}{32}\left[\frac{2}{5}(25\sqrt{5} - 1) - \frac{2}{3}\cdot(5\sqrt{5} - 1)\right]$$

$$= \frac{1}{32}\left[10\sqrt{5} - \frac{2}{5} - \frac{10}{3}\sqrt{5} + \frac{2}{3}\right] = \frac{1}{32}\left[\frac{30\sqrt{5}}{3} - \frac{10\sqrt{5}}{3} - \frac{6}{15} + \frac{10}{15}\right]$$

$$= \frac{1}{32}\left[\frac{20\sqrt{5}}{3} + \frac{4}{15}\right]$$ Put back into main integral.

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Should eventually get $\frac{5\sqrt{5}}{48} + \frac{1}{240}$

(We did not work out these details during class)

We had

$$\iint_S yz\,dS = \int_0^{\pi/2} \sin v \cos v \underbrace{\int_0^1 u^3\sqrt{1+4u^2}\,du}_{I_1}\,dv$$

, and we calculated that

$$I_1 = \int_0^1 u^3\sqrt{1+4u^2} = \frac{1}{32}\left[\frac{20\sqrt{5}}{3} + \frac{4}{15}\right]$$

So,

$$\iint_S yz\,dS = \int_0^{\pi/2} \sin v \cos v\,(I_1)\,dv = \int_0^{\pi/2} \underbrace{\sin v}_{t}\,\underbrace{\cos v \left(\frac{1}{32}\left[\frac{20\sqrt{5}}{3} + \frac{4}{15}\right]\right) dv}_{dt}$$

$$= \frac{1}{32}\left[\frac{20\sqrt{5}}{3} + \frac{4}{15}\right] \int_{v=0}^{v=\frac{\pi}{2}} t\,dt$$

$$= \frac{1}{32}\left[\frac{100\sqrt{5}+4}{15}\right] \cdot \frac{t^2}{2}\Bigg|_{v=0}^{v=\pi/2}$$

$$= \frac{100\sqrt{5}+4}{480} \cdot \frac{\sin^2 v}{2}\Bigg|_{v=0}^{v=\pi/2}$$

$$= \frac{100\sqrt{5}+4}{960}\left[\sin^2\left(\frac{\pi}{2}\right) - \sin^2(0)\right] = \frac{100\sqrt{5}+4}{960}\left[1^2 - 0^2\right]$$

$$= \frac{\cancel{2}^{\,1}(50\sqrt{5}+2)}{\cancel{960}_{\,480}} = \boxed{\frac{50\sqrt{5}+2}{480}}$$

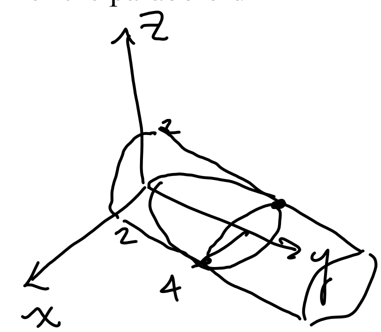
$$= \frac{50\sqrt{5}}{480} + \frac{2}{480} = \frac{5\sqrt{5}}{48} + \frac{1}{240}$$

Let $t = \sin v$

$\frac{dt}{dv} = \cos v$

$dt = \cos v\,dt$

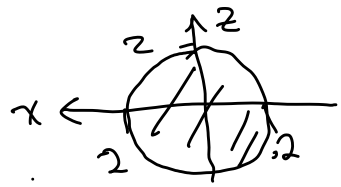
Example 3: Evaluate the surface integral $\iint_S y\,dS$, where S is the portion of the paraboloid $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 4$.



Where do they intersect?

Setting $\underbrace{y = x^2 + z^2 = x^2 + z^2 = 4}_{y=4}$:

Here, we project onto xz-plane to get region R:



$x^2 + z^2 = 4$
circle radius 2

Parametrize the surface:

Let ~~x~~ y

$$\left.\begin{array}{l} x = r\cos\theta \\ z = r\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} x = u\cos v \\ z = u\sin v \end{array}$$

$$x^2 + z^2 = y$$
$$(u\cos v)^2 + (u\sin v)^2 = y$$
$$u^2\cos^2 v + u^2\sin^2 v = y$$
$$u^2 = y$$

$\vec{r}(u,v) = \langle u\cos v, u^2, u\sin v\rangle \longrightarrow$ $0 \le u \le 2$, $0 \le v \le 2\pi$

$\vec{r}_u(u,v) = \langle \cos v, 2u, \sin v\rangle$

$\vec{r}_v(u,v) = \langle -u\sin v, 0, u\cos v\rangle$

$$(\vec{r}_u \times \vec{r}_v)(u,v) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & 2u & \sin v \\ -u\sin v & 0 & u\cos v \end{vmatrix}$$

$$= \langle 2u^2\cos v - 0, -u\sin^2 v - u\cos^2 v, 0 + 2u^2\sin v\rangle$$

$$= \langle 2u^2\cos v, -u, 2u^2\sin v\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^4\cos^2 v + (-u)^2 + 4u^4\sin^2 v} = \sqrt{4u^4 + u^2} = \sqrt{u^2}\sqrt{4u^2+1} = u\sqrt{4u^2+1}$$

$\boxed{\text{Note: } u \geq 0, \text{ so } \sqrt{u^2} = u}$

$$\iint_S y\,dS = \iint_R \underbrace{u^2}_{y} \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^{2\pi}\int_0^2 u^2 \cdot u\sqrt{4u^2+1}\,du\,dv$$

see next page

eventually you get $\frac{391\pi\sqrt{17}}{60} + \frac{\pi}{60}$

Example 3 continued: (we did not work out these details during class)

$$\iint_S y\,dS = \iint_R \underbrace{u^2}_{y} \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^{2\pi}\int_0^2 \underbrace{u^2}_{\frac{1}{4}(t-1)} \cdot \underbrace{u\underbrace{\sqrt{4u^2+1}}_{t^{1/2}}\,du}_{\frac{1}{8}dt}\,dv$$

$$= \int_0^{2\pi}\int_{t=1}^{t=17} \frac{1}{4}(t-1)\cdot\frac{1}{8}\,t^{1/2}\,dt\,dv$$

$$= \int_0^{2\pi}\int_1^{17} \frac{1}{32}\left(t^{3/2}-t^{1/2}\right)dt\,dv$$

$$= \frac{1}{32}\int_0^{2\pi}\left[\frac{t^{5/2}}{5/2} - \frac{t^{3/2}}{3/2}\right]\Bigg|_{1=t}^{17=t} dv$$

$$= \frac{1}{32}\int_0^{2\pi}\left[\frac{2}{5}t^{5/2}\Big|_1^{17} - \frac{2}{3}t^{3/2}\Big|_1^{17}\right]dv$$

$$= \frac{1}{32}\int_0^{2\pi}\left[\frac{2}{5}\left(17^{5/2}-1^{5/2}\right) - \frac{2}{3}\left(17^{3/2}-1^{3/2}\right)\right]dv$$

$$= \frac{1}{32}\left[\frac{2}{5}\left(289\sqrt{17}-1\right) - \frac{2}{3}\left(17\sqrt{17}-1\right)\right]\int_0^{2\pi} dv$$

$$= \left[\frac{289\sqrt{17}}{80} - \frac{1}{80} - \frac{1}{48}\cdot 17\sqrt{17} + \frac{1}{48}\right]\theta\Big|_0^{2\pi}$$

$$= \left[\frac{289\sqrt{17}}{80} - \frac{1}{80} + \frac{17\sqrt{17}}{48} + \frac{1}{48}\right](2\pi - 0)$$

$$= \frac{289\pi\sqrt{17}}{40} - \frac{\pi}{40} + \frac{17\pi\sqrt{17}}{24} + \frac{\pi}{24}$$

$$= \frac{867\pi\sqrt{17}}{120} - \frac{3\pi}{120} + \frac{85\sqrt{17}}{120} + \frac{5\pi}{120}$$

$$= \frac{782\pi\sqrt{17}}{120} + \frac{2\pi}{120} = \boxed{\frac{391\pi\sqrt{17}+\pi}{60}}$$

$t = 4u^2+1$

$\frac{dt}{du} = 8u$

$dt = 8u\,du$

$\frac{1}{8}dt = u\,du$

$t-1 = 4u^2$

$\frac{1}{4}(t-1) = u^2$

$u=0 \Rightarrow t = 4(0)^2+1 = 1$

$u=2 \Rightarrow t = 4(2)^2+1 = 17$

Oriented surfaces:

Definition: Orientable

A surface S is called *orientable* when a unit normal vector $\mathbf{N}$ can be defined at every nonboundary point of S in such a way that the normal vectors vary continuously over the surface S.

When we *orient* a surface, we choose one of the two possible unit normal vectors at a point, and then all the other normal vectors are selected so they vary continuously over the surface. In order to be orientable, the surface must have two distinct sides. For a closed surface (or a surface in which one side can be considered the "outside" and the other the "inside", it is customary to choose the unit normal vector that points outward.

For an orientable surface defined by $z = g(x, y)$, we write $G(x, y, z) = z - g(x, y)$. Then, the surface can be oriented by either of these two normal vectors:

$$\mathbf{N}_1 = \frac{\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|} \qquad \text{or} \qquad \mathbf{N}_2 = \frac{-\nabla G(x, y, z)}{\|\nabla G(x, y, z)\|}$$

For an orientable surface defined parametrically by by $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$, the surface can be oriented by either of these two normal vectors:

$$\mathbf{N}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \qquad \text{or} \qquad \mathbf{N}_2 = \frac{\mathbf{r}_v \times \mathbf{r}_u}{\|\mathbf{r}_v \times \mathbf{r}_u\|}$$

Surface integrals of vector fields:

Definition: Flux

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{N}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS .$$

This is also called the *flux* of $\mathbf{F}$ across S.

<u>Theorem</u>: Evaluating a Flux Integral

Let S be an oriented surface given by $z = g(x, y)$ and let R be its projection onto the xy-plane. Then,

$$\iint \vec{F} \cdot \nabla G \, dS$$

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_R \mathbf{F} \cdot \left[-g_x(x, y)\,\mathbf{i} - g_y(x, y)\,\mathbf{j} + \mathbf{k} \right] dA \quad \text{(oriented upward)}$$

and $$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_R \mathbf{F} \cdot \left[g_x(x, y)\,\mathbf{i} + g_y(x, y)\,\mathbf{j} - \mathbf{k} \right] dA \quad \text{(oriented downward).}$$

If an oriented surface S is given by $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$, defined on a domain D in the uv-plane, then

$$\iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.$$

If we have $z = g(x,y)$, write $G(x,y,z) = z - g(x,y) = 0$

Function form of z

$$N \, dS = \frac{\nabla G(x,y,z)}{\|\nabla G(x,y,z)\|} \, dS = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \, dS$$

$$= \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{(g_x)^2 + (g_y)^2 + 1}} \sqrt{1 + g_x^2 + g_y^2} \, dA$$

Recall:

$$dS = \sqrt{1 + g_x^2 + g_y^2} \, dA$$

$$dS = \|\nabla G(x,y,z)\| \, dA$$

$$dS = \|\vec{r}_u \times \vec{r}_v\| \, dA$$

$$= \langle -g_x, -g_y, 1 \rangle \, dA = \nabla G \, dA$$

Parametric

$$\vec{N} \, dS = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \, dS = \frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}} \cdot \cancel{\|\vec{r}_u \times \vec{r}_v\|} \, dA$$

$$= (\vec{r}_u \times \vec{r}_v) \, dA$$

Example 4: Find the flux of $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$ across the upward oriented portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$.

$$G(x,y,z) = z - 4 + x^2 + y^2 = 0$$

$$\nabla G(x,y,z) = \langle 2x, 2y, 1\rangle$$

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iint_R \langle xy, yz, xz\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$$

$$= \iint_R (2x^2y + 2y^2z + xz)\, dA \qquad z = 4 - x^2 - y^2$$

$$= \int_0^1\int_0^1 (2x^2y + 2y^2(4 - x^2 - y^2) + x(4 - x^2 - y^2)\, dy\, dx$$

$$= \int_0^1\int_0^1 (2x^2y + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2)\, dy\, dx$$

$$= \int_0^1 \left[x^2y^2 - \frac{2x^2y^3}{3} - \frac{2y^5}{5} + 4xy - x^3y - \frac{xy^3}{3} + \frac{8y^3}{3}\right)\Big|_0^1 dx$$

$$= \int_0^1 \left[x^2 + \frac{8}{3} - \frac{2}{3}x^2 - \frac{2}{5} + 4x - x^3 - \frac{x}{3} - 0\right] dx$$

$$= \int_0^1 \left(-x^3 + \frac{1}{3}x^2 + \frac{11}{3}x + \frac{40}{15} - \frac{6}{15}\right] dx$$

$$= -\frac{x^4}{4} + \frac{1}{3}\cdot\frac{x^3}{3} + \frac{11}{3}\frac{x^2}{2} + \frac{34}{15}x \Big|_0^1$$

$$= -\frac{1}{4} + \frac{1}{9} + \frac{11}{6} + \frac{34}{15} - 0 \qquad = \boxed{\frac{713}{180}}$$

Example 5: Find the flux of $\mathbf{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ across the upward oriented helicoid defined by $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

$$\vec{F}(x,y,z) = \langle y,\ x,\ z^2 \rangle$$

$$\vec{F}(u,v) = \langle u\sin v,\ u\cos v,\ v^2 \rangle$$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v \rangle$$

$$\vec{r}_u(u,v) = \langle \cos v,\ \sin v,\ 0 \rangle$$

$$\vec{r}_v(u,v) = \langle -u\sin v,\ u\cos v,\ 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v - 0,\ -\cos v,\ u\cos^2 v + u\sin^2 v \rangle$$

$$= \langle \sin v,\ -\cos v,\ u \rangle$$

$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iint_R \vec{F}(u,v) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$$= \int_0^{\pi}\int_0^1 \langle u\sin v,\ u\cos v,\ v^2 \rangle \cdot \langle \sin v, -\cos v, u \rangle\, du\, dv$$

Recall:

$$\begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ -\cos 2v = \sin^2 v - \cos^2 v \end{cases}$$

$$= \int_0^{\pi}\int_0^1 \left[u\sin^2 v - u\cos^2 v + uv^2\right] du\, dv$$

$$= \int_0^{\pi}\int_0^1 \left[u\underbrace{(\sin^2 v - \cos^2 v)}_{-\cos 2v} + uv^2\right] du\, dv$$

$$= \int_0^{\pi}\int_0^1 \left[-u\cos 2v + uv^2\right] du\, dv = \int_0^{\pi} \left[-\frac{u^2}{2}\cos 2v + \frac{u^2}{2}\cdot v^2\right]\Big|_0^1 dv$$

$$= \int_0^{\pi} \left[-\frac{1}{2}\cos 2v + \frac{1}{2}v^2 + 0 - 0\right] dv$$

$$= \left[-\frac{1}{2}\cdot\frac{1}{2}\sin 2v + \frac{1}{2}\cdot\frac{v^3}{3}\right]\Big|_0^{\pi}$$

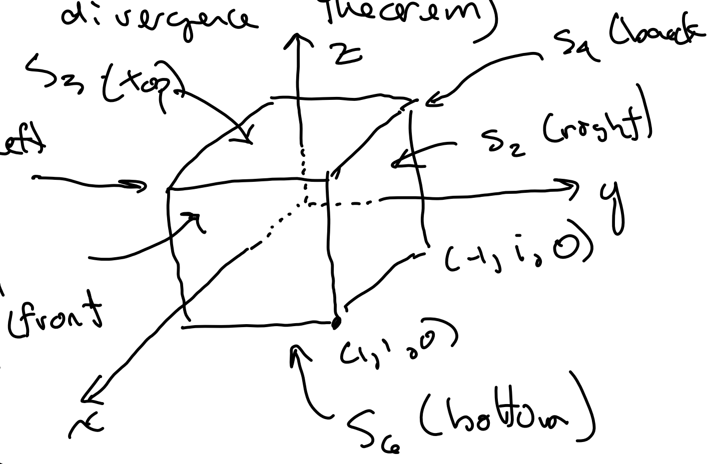
$$= -\frac{1}{4}\cancelto{0}{\sin 2\pi} + \frac{1}{4}\cancelto{0}{\sin(2(0))} + \frac{1}{6}(\pi^3 - 0^3)$$

$$= \boxed{\frac{\pi^3}{6}}$$

Example 6: Find the outward flux of $\mathbf{F}(x,y,z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$ across the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

Calculating flux directly, using a surface integral:

(instead of using the divergence theorem)



We need to find $\int \vec{F} \cdot \vec{N}\, dS$ for each surface, then add them.

$\vec{F}(x,yz) = \langle x, 2y, 3z \rangle$

S_1: $\vec{F}(1,y,z) = \langle 1, 2y, 3z \rangle$, $\vec{N}_1 = \langle 1,0,0 \rangle$

S_2: $\vec{F}(x,1,z) = \langle x, 2, 3z \rangle$, $\vec{N}_2 = \langle 0,1,0 \rangle$

S_3: $\vec{F}(x,y,1) = \langle x, 2y, 3 \rangle$, $N_3 = \langle 0,0,1 \rangle$

S_4: $\vec{F}(-1,y,z) = \langle -1, 2y, 3z \rangle$, $N_4 = \langle -1,0,0 \rangle$

S_5: $\vec{F}(x,-1,z) = \langle x, -2, 3z \rangle$, $N_5 = \langle 0,-1,0 \rangle$

S_6: $\vec{F}(x,y,-1) = \langle x, 2y, -3z \rangle$, $N_6 = \langle 0,0,-1 \rangle$

$$I_1 = \int_{S_1} \vec{F}_1 \cdot \vec{N}_1\, dS = \iint_{R_1} \langle 1, 2y, 3z \rangle \cdot \langle 1,0,0 \rangle\, dA = \int_{-1}^{1}\int_{-1}^{1} 1\, dy\, dz$$

$$= \int_{-1}^{1} y\Big|_{-1}^{1} dz = \int_{-1}^{1} (1-(-1))\,dz = \int_{-1}^{1} 2\,dz = 2z\Big|_{-1}^{1} = 2(1-(-1)) = 2(2) = 4$$

$$I_2 = \int_{-1}^{1}\int_{-1}^{1} \langle x, 2, 3z \rangle \cdot \langle 0,1,0 \rangle\, dx\, dz = \int_{-1}^{1}\int_{-1}^{1} 2\,dx\,dz = \int_{-1}^{1} 2x\Big|_{-1}^{1} dz = \int_{-1}^{1} 2(1-(-1))\,dz$$

$$= \int_{-1}^{1} 4\,dz = 4z\Big|_{-1}^{1} = 4(2) = 8$$

Similarly, $I_3 = 12$, $I_4 = 4$, $I_5 = 8$, $I_6 = 12$

$$\text{Outward Flux} = \int_S \vec{F} \cdot N\, dS = I_1 + I_2 + \dots + I_6 = 4 + 8 + 12 + 4 + 8 + 12$$

$$= 24 + 24 = \boxed{48}$$

In 15.6 notes

Summary:

Line Integrals:

$$ds = \|\mathbf{r}'(t)\|\,dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt \qquad \text{(scalar-valued function)}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big)\cdot\mathbf{r}'(t)\,dt \qquad \text{(vector field)}$$

Surface Integrals: (surface defined by $z = g(x,y)$) $\quad G(x,y,z) = z - g(x,y)$

$$dS = \sqrt{1 + [g_x(x,y)]^2 + [g_y(x,y)]^2}\,dA = \|\nabla G(x,y,z)\|\,dA$$

$$\iint_S f(x,y,z)\,dS = \iint_R f\big(x,y,g(x,y)\big)\sqrt{1 + [g_x(x,y)]^2 + [g_y(x,y)]^2}\,dA \qquad \text{(scalar-valued function)}$$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_R \mathbf{F}\cdot\big[-g_x(x,y)\,\mathbf{i} - g_y(x,y)\,\mathbf{j} + \mathbf{k}\big]\,dA = \iint_R \vec{F}\cdot\nabla G\,dA \qquad \text{(vector field)}$$

Surface Integrals: (surface defined by parametrically by $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$)

$$dS = \|\mathbf{r}_u(u,v)\times\mathbf{r}_v(u,v)\|\,dA$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\,\|\mathbf{r}_u\times\mathbf{r}_v\|\,dA \qquad \text{(scalar-valued function)}$$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA \qquad \text{(vector field)}$$