15.7: Divergence Theorem

Recall: Green's Theorem:

If R is a simply connected region in $\mathbb{R}^2$ with a positively oriented piecewise smooth boundary C, and if $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j}$ is a vector field with M and N having continuous first partial derivatives, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA\,.$$

We can rewrite Green's Theorem in two alternative forms:

Alternative Form #1: $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\;dA$

Alternative Form #2: $\displaystyle\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \iint_R \operatorname{div}\mathbf{F}\;dA$

Alternative Form #1 requires us to rewrite the vector field $\mathbf{F}$ in $\mathbb{R}^3$ as $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + 0\,\mathbf{k}$. Alternative Form #2 requires us to begin with the arc length parameter version of the line integral, $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r}$, and apply it to $\mathbf{N}$ instead of $\mathbf{T}$.

If we extend Alternative Green's #1 by an extra dimension, to a curve C in $\mathbb{R}^3$, we end up with Stokes' Theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{N}\, dA \qquad \text{(coming up in Section 15.8)}$$

Note the right-hand side is a surface integral. (In Green's, the right-hand side is an area integral.)

If we extend Alternative Green's #2 by an extra dimension, we end up with the Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{N}\, dS = \iiint_Q \operatorname{div}\mathbf{F}\, dV \qquad \text{(covered here in Section 15.7)}$$

Note that this converts a surface integral to a volume integral. (Green's converts a line integral to an area integral.)

> Divergence Theorem:
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> Let Q be a solid region bounded by a closed surface S oriented by an outwardly directed unit normal vector $\mathbf{N}$. If $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field with M, N, and P having continuous first partial derivatives in Q, then
>
> $$\iint_S \mathbf{F}\cdot\mathbf{N}\, dS = \iiint_Q \operatorname{div}\mathbf{F}\, dV\ .$$

Example 1: Find the outward flux of $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ across the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

Example 2: Find the outward flux of $\mathbf{F}(x,y,z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$ across the cube with vertices $(\pm 1, \pm 1, \pm 1)$. (Same as Section 15.6, Example 6).

Example 3: Find the outward flux of $\mathbf{F}(x,y,z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$ across the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.