15.8: Stokes' Theorem

Recall: Alternative Version #1 of Green's Theorem:

If R is a simply connected region in $\mathbb{R}^2$ with a positively oriented piecewise smooth boundary C, and if $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + 0\mathbf{k}$ is a vector field with M and N having continuous first partial derivatives, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA.$$

If we extend this result to a curve C in $\mathbb{R}^3$, we end up with Stokes' Theorem.

Stokes' Theorem:

Let S be an oriented surface S with outwardly directed unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple closed curve C with a positive orientation. If $\mathbf{F}(x, y, z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field with M, N, and P having continuous first partial derivatives on an open region containing S and C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{N}\, dA.$$

Note: The orientation of the boundary curve C and the normal vector $\mathbf{N}$ follow the "right-hand rule." If your right thumb points in the direction of the normal vector, then your fingers curl in the direction of the boundary orientation (and vice versa).

Also Note: Parallelism between Stokes' Theorem and the Fundamental Theorem of Calculus. (In both, the value of the integral depends only on the values on the boundary.)

<u>Example 1:</u> Verify that Stokes' Theorem holds for $\mathbf{F}(x, y, z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$.

<u>Example 2:</u> Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^{-x}\,\mathbf{i} + e^{x}\,\mathbf{j} + e^{z}\,\mathbf{k}$, and C is the boundary of the first-octant portion of the plane $2x + y + 2x = 2$.