15.8: Stokes' Theorem

Recall: Alternative Version #1 of Green's Theorem:

If R is a simply connected region in $\mathbb{R}^2$ with a positively oriented piecewise smooth boundary C, and if $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + 0\mathbf{k}$ is a vector field with M and N having continuous first partial derivatives, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (\operatorname{curl}\mathbf{F})\cdot\mathbf{k}\, dA.$$

If we extend this result to a curve C in $\mathbb{R}^3$, we end up with Stokes' Theorem.

> Stokes' Theorem:
>
> Let S be an oriented surface S with outwardly directed unit normal vector $\mathbf{N}$, bounded by a piecewise smooth simple closed curve C with a positive orientation. If $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is a vector field with M, N, and P having continuous first partial derivatives on an open region containing S and C, then
>
> $$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{N}\, d\cancel{A}.\ dS$$

Note: The orientation of the boundary curve C and the normal vector $\mathbf{N}$ follow the "right-hand rule." If your right thumb points in the direction of the normal vector, then your fingers curl in the direction of the boundary orientation (and vice versa).

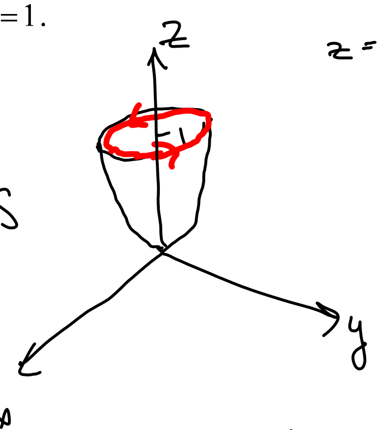
Also Note: Parallelism between Stokes' Theorem and the Fundamental Theorem of Calculus. (In both, the value of the integral depends only on the values on the boundary.)

Fun Theorem of Calculus: If F is an antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$

Example 1: Verify that Stokes' Theorem holds for $\mathbf{F}(x,y,z) = y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 1$.

$\vec{F}(x,y,z) = \langle y^2, x, z^2 \rangle$

We must show that $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{N}\,dS$



$z = 1 \Rightarrow x^2+y^2=1$
circle
radius 1

(a) $\iint_S (\text{curl}\,\vec{F})\cdot\vec{N}\,dS$

$= \iint_R \langle 0,0,1-2y\rangle\cdot\langle -2x,-2y,1\rangle\,dA$

$= \int_0^{2\pi}\int_0^1 (1-2y)(1)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (1-2r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (r-2r^2\sin\theta)\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{2r^3}{3}\sin\theta\right]\Big|_0^1\,d\theta$

$= \int_0^{2\pi}\left[\frac{1}{2}-\frac{2(1)^3}{3}\sin\theta - 0 - 0\right]d\theta$

$= \frac{1}{2}\theta\Big|_0^{2\pi} + \frac{2}{3}\cos\theta\Big|_0^{2\pi} = \frac{1}{2}(2\pi-0)+\frac{2}{3}(\cos 2\pi - \cos 0)$

$= \pi + \frac{2}{3}\underbrace{(1-1)}_{0} = \pi + 0 = \boxed{\pi}$

$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x & z^2 \end{vmatrix}$

$= \langle 0-0,\ 0-0,\ 1-2y\rangle$

$= \langle 0, 0, 1-2y\rangle$

$z = x^2+y^2$

$G(x,y,z) = z - x^2 - y^2$

$\nabla G(x,y,z) = \langle -2x, -2y, 1\rangle$

(b) $\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle y^2, x, z^2\rangle\cdot\langle -\sin t, \cos t, 0\rangle\,dt$

$= \int_0^{2\pi} \langle \sin^2 t, \cos t, 1^2\rangle\cdot\langle -\sin t, \cos t, 0\rangle\,dt$

$= \int_0^{2\pi} (-\sin^3 t + \cos^2 t + 0)\,dt$

$= \int_0^{2\pi} (\cos^2 t - \sin^3 t)\,dt = \int_0^{2\pi}\Big(\cos^2 t - \sin t\,\underbrace{(1-\cos^2 t)}_{\sin^2 t}\Big)\,dt$

$= \int_0^{2\pi} (\cos^2 t - \sin t + \sin t\cos^2 t)\,dt$

see next page

Parametrize C:

$x = 1\cos t$

$y = 1\sin t$

$z = 1$

$\vec{r}(t) = \langle \cos t, \sin t, 1\rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0\rangle$

$0 \le t \le 2\pi$

Ex2 cont'd:

$$\int_0^{2\pi}\left[\frac{1}{2}(1+\cos 2t)\right]dt - \int_0^{2\pi}\sin t\,dt + \int_0^{2\pi}\sin t\cos^2 t\,dt$$

$$= \frac{1}{2}t\Big|_0^{2\pi} + \frac{1}{2}\cdot\frac{1}{2}\sin 2t\Big|_0^{2\pi} + \cos t\Big|_0^{2\pi} + \frac{1}{-1}\cdot\frac{\cos^3 t}{3}\Big|_0^{2\pi}$$

$u = \cos t$

$du = -\sin t\,dt$

$\frac{du}{-1} = \sin t\,dt$

$$= \frac{1}{2}(2\pi - 0) + \frac{1}{4}\cancel{\sin 4\pi}^{0} - \frac{1}{4}\cancel{\sin 0}^{0} + \cos 2\pi - \cos 0 - \frac{1}{3}[\cos^3 2\pi] + \frac{1}{3}\cos^3(0)$$

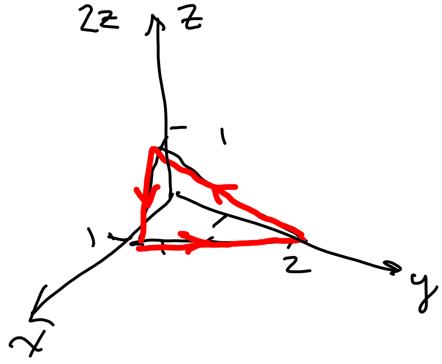
$$= \pi + 1 - 1 - \frac{1}{3}(1)^3 + \frac{1}{3}(1)^3$$

$$= \boxed{\pi}$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{N}\,dS = \pi.$$

$\therefore$ Stokes' Theorem Holds.

Example 2: Use Stokes' Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=e^{-x}\,\mathbf{i}+e^{x}\,\mathbf{j}+e^{z}\,\mathbf{k}$, and C is the boundary of the first-octant portion of the plane $2x+y+2z=2$.



Stokes: $\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-x} & e^{x} & e^{z} \end{vmatrix}$$

$= \langle 0-0,\ 0-0,\ e^{x}-0\rangle$

$= \langle 0, 0, e^{x}\rangle$

Plane: $2x+y+2z=2$

$x,y=0 \Rightarrow z=1$

$x,z=0 \Rightarrow y=2$

$y,z=0 \Rightarrow x=1$

$2x+y+2z=2$

$2x+y+2z-2=0$

Divide by 2:

$x+\frac{1}{2}y+z-1=0$

$G(x,y,z) = x+\frac{1}{2}y+z-1$

$\nabla G(x,y,z) = \langle 1, \frac{1}{2}, 1\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{N}\,dS$$

$$= \iint_R \langle 0,0,e^{x}\rangle\cdot\langle 1,\tfrac{1}{2},1\rangle\,dA$$

$$= \int_0^2\int_0^{-\frac{1}{2}y+1} e^{x}\,dx\,dy$$

$$= \int_0^2 e^{x}\Big|_0^{-\frac{1}{2}y+1}\,dy$$

$$= \int_0^2\left(e^{-\frac{1}{2}y+1}-e^{0}\right)dy$$

$$= \int_0^2\left(e^{-\frac{1}{2}y+1}-1\right)dy$$

$$= \int_0^2 e^{-\frac{1}{2}y+1}\,dy - \int_0^2 1\,dy$$

$u=-\frac{1}{2}y+1$

$\frac{du}{dy}=-\frac{1}{2}$

$du=-\frac{1}{2}dy$

$-2\,du=dy$

$$= -2e^{-\frac{1}{2}y+1}\Big|_0^2 - y\Big|_0^2$$

$$= -2\left[e^{-1+1}-e^{0+1}\right]-2+0 = -2\left[e^{0}-e\right]-2$$

$$= -2\left[1-e\right]-2 = -2+2e-2 = \boxed{2e-4}$$

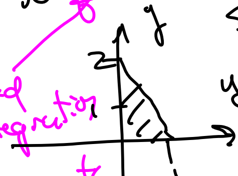
Note: What if I use $dy\,dx$

$$\int_0^1\int_0^{-2x+2} e^{x}\,dy\,dx$$

$$= \int_0^1 e^{x}y\Big|_0^{-2x+2}\,dx$$

$$= \int_0^1 e^{x}(-2x+2-0)\,dx$$

need integration by parts



Slope $= -\frac{2}{1}$

$y=-2x+2$

$y-2=-2x$

$\frac{y-2}{-2}=x$

$-\frac{1}{2}y+1=x$