11.7: Cylindrical and Spherical Coordinates

Cylindrical coordinates:

Cylindrical coordinates extend the polar coordinate system into $\mathbb{R}^3$.

$$\underset{\text{Rectangular}}{P(x,y,z)} \longrightarrow \underset{\text{Cylindrical}}{P'(r,\theta,z)}$$

In a cylindrical coordinate system, a point *P* in $\mathbb{R}^3$ is represented by an ordered triple (r,θ,z) .

1. (r,θ) is a polar representation of the projection of *P* in the *xy*-plane.
2. z has the same meaning as in rectangular coordinates.

Note: Cylindrical coordinates are especially useful for representing surfaces for which the *z*-axis is the axis of symmetry (cylindrical surfaces and surfaces of revolution).

Converting between rectangular and cylindrical coordinate systems:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Example 1: Convert the point with rectangular coordinates $(-2\sqrt{2}, 2\sqrt{2}, 2)$ to cylindrical coordinates.

Example 2: Convert the point with cylindrical coordinates $\left(3, \frac{\pi}{3}, -5\right)$ to rectangular coordinates.

Example 3: Convert the equation $z = x^2 + y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Example 4: Convert the equation $z = x^2 - y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Example 5: Convert the equation $4 = x^2 + y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Example 6: Convert the equation $x^2 + y^2 = z^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Example 7: Convert the equation $r = 2\cos\theta$ in cylindrical coordinates into an equation in rectangular coordinates.

Spherical coordinates:

$$\underset{\text{Rectangular}}{P(x, y, z)} \longrightarrow \underset{\text{Spherical}}{P'(\rho, \theta, \phi)}$$

In a spherical coordinate system, a point P in $\mathbb{R}^3$ is represented by an ordered triple (ρ, θ, ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the angle between the positive x-axis and the projection of $\overrightarrow{OP}$ in the xy-plane (same θ as in cylindrical coordinates).
3. ϕ is the angle between the positive z-axis and $\overrightarrow{OP}$ ($0 \leq \phi \leq \pi$)

<u>Note</u>: Spherical coordinates are especially useful for surfaces that are symmetric about a point, or center.

<u>Converting between rectangular and spherical coordinate systems</u>:

$x = \rho \sin\phi \cos\theta$ $\qquad$ $y = \rho \sin\phi \sin\theta$ $\qquad$ $z = \rho \cos\phi$

$x^2 + y^2 + z^2 = \rho^2$ $\qquad$ $\tan\theta = \dfrac{y}{x}$ $\qquad$ $\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

<u>Converting between cylindrical and spherical coordinate systems</u> $r \geq 0$:

$r^2 = \rho^2 \sin^2\phi$ $\qquad$ $\theta = \theta$ $\qquad$ $z = \rho \cos\phi$

$\rho = \sqrt{r^2 + z^2}$ $\qquad$ $\theta = \theta$ $\qquad$ $\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$

<u>Example 8:</u> Convert the equation $\rho = c$ (c a constant) in spherical coordinates into an equation in rectangular coordinates.

<u>Example 9:</u> Convert the equation $\phi = c$, $0 < c < \dfrac{\pi}{2}$, in spherical coordinates into an equation in rectangular coordinates.

Example 10: Convert the equation $x^2 + y^2 - 3z^2 = 0$ to spherical coordinates from rectangular coordinates.

Example 11: Describe the surface with equation $\phi = \frac{\pi}{2}$ in spherical coordinates.

Example 12: Describe the surface with equation $\phi = \frac{\pi}{4}$ in spherical coordinates.

Example 13: Convert the point that is represented by $(1, 2, 3)$ in rectangular coordinates to cylindrical and spherical coordinates.