11.7: Cylindrical and Spherical Coordinates

Cylindrical coordinates:

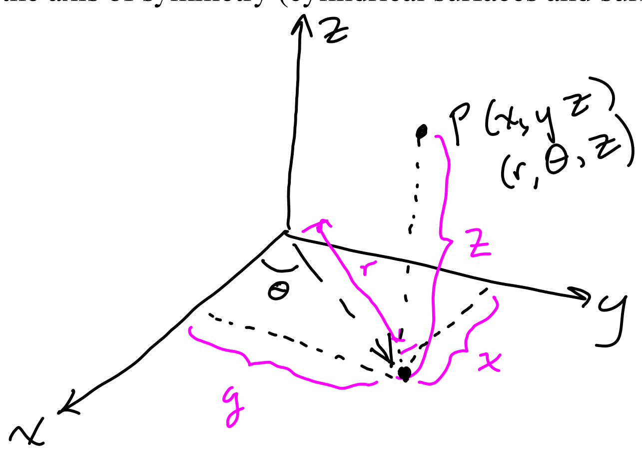
Cylindrical coordinates extend the polar coordinate system into $\mathbb{R}^3$.

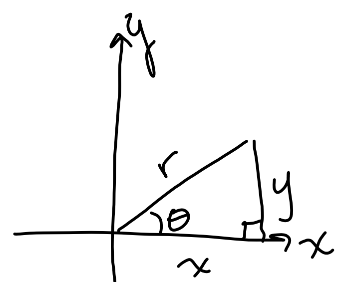
$$\underset{\text{Rectangular}}{P(x,y,z)} \longrightarrow \underset{\text{Cylindrical}}{P'(r,\theta,z)}$$

In a cylindrical coordinate system, a point P in $\mathbb{R}^3$ is represented by an ordered triple (r,θ,z).

1. (r,θ) is a polar representation of the projection of P in the xy-plane.
2. z has the same meaning as in rectangular coordinates.

Note: Cylindrical coordinates are especially useful for representing surfaces for which the z-axis is the axis of symmetry (cylindrical surfaces and surfaces of revolution).





$x = r\cos\theta$
$y = r\sin\theta$

Converting between rectangular and cylindrical coordinate systems:

$x = r\cos\theta$	$y = r\sin\theta$	$z = z$
$x^2 + y^2 = r^2$	$\tan\theta = \dfrac{y}{x}$	$z = z$

Example 1: Convert the point with rectangular coordinates $(-2\sqrt{2}, 2\sqrt{2}, 2)$ to cylindrical coordinates.

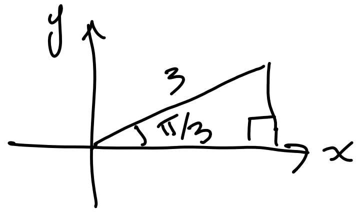
$x = -2\sqrt{2},\ y = 2\sqrt{2},\ z = 2$

$x^2 + y^2 = r^2$
$(-2\sqrt{2})^2 + (2\sqrt{2})^2 = r^2$
$8 + 8 = r^2$
$16 = r^2$
$r = \pm 4$, choose $r = 4$

$\tan\theta = \frac{y}{x} = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1$

$\theta = -\frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$

$\boxed{(r,\theta,z) = (4, \frac{3\pi}{4}, 2)}$ or $\boxed{(r,\theta,z) = (4, -\frac{\pi}{4}, 2)}$



Example 2: Convert the point with cylindrical coordinates $\left(3, \frac{\pi}{3}, -5\right)$ to rectangular coordinates.

$x = r\cos\theta = 3\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$y = r\sin\theta = 3\sin\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

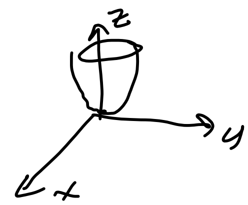
$$\boxed{(x, y, z) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, -5\right)}$$

Example 3: Convert the equation $z = x^2 + y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Circular Paraboloid centered on z-axis

$z = x^2 + y^2$

$\boxed{z = r^2}$ [using $x^2 + y^2 = r^2$]



Example 4: Convert the equation $z = x^2 - y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

$z = x^2 - y^2$

$z = (r\cos\theta)^2 - (r\sin\theta)^2$

$z = r^2\cos^2\theta - r^2\sin^2\theta$

$z = r^2(\cos^2\theta - \sin^2\theta)$

$\boxed{z = r^2\cos(2\theta)}$

Traces:

xy: $0 = x^2 - y^2$

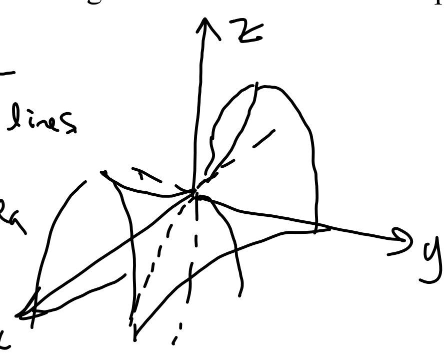
$y = \pm x$ lines

yz: $z = -y^2$

parabola

xz: $z = x^2$

parabola

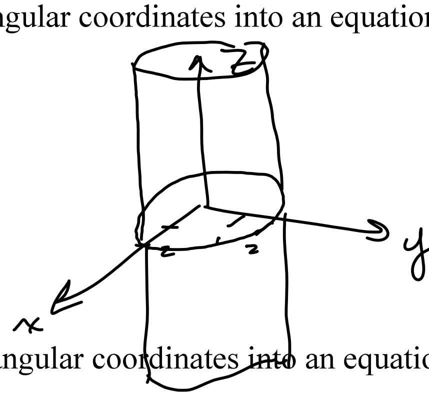


Example 5: Convert the equation $4 = x^2 + y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

$4 = r^2$

$\boxed{r = 2}$

It's a cylinder of radius 2.



Example 6: Convert the equation $x^2 + y^2 = z^2$ in rectangular coordinates into an equation in cylindrical coordinates.

$r^2 = z^2$

$r = z$

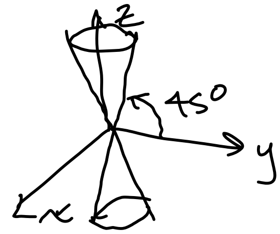
xy: $x^2 + y^2 = 0$ Pt $(0,0,0)$

yz: $y^2 = z^2$

$y = \pm z$ line

xz: $x = \pm z$

Circular Cone:

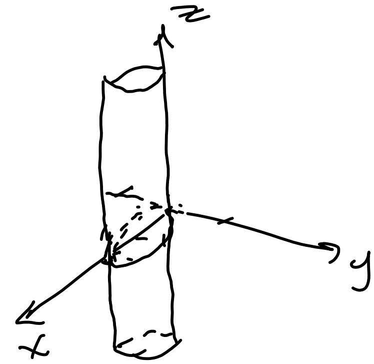


Example 7: Convert the equation $r = 2\cos\theta$ in cylindrical coordinates into an equation in rectangular coordinates.

$r = 2\cos\theta$

Multiply both sides by r: $r^2 = 2r\cos\theta$

$$\boxed{x^2 + y^2 = 2x}$$

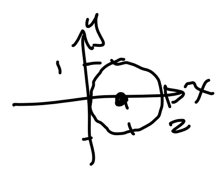


To sketch it, complete the square:

$x^2 - 2x + y^2 = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$



Spherical coordinates:

$P(x, y, z) \longrightarrow P'(\rho, \theta, \phi)$ (rho, theta, phi)

Rectangular → Spherical

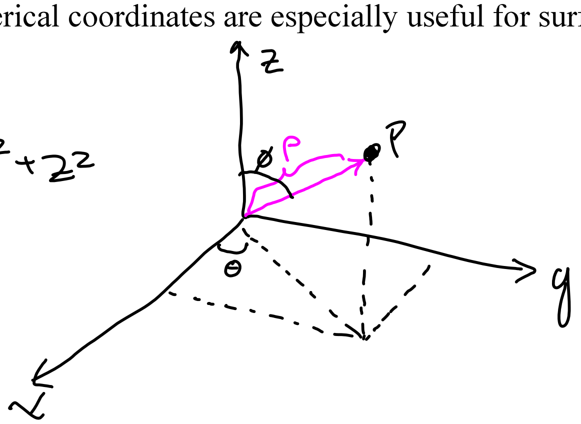
In a spherical coordinate system, a point P in $\mathbb{R}^3$ is represented by an ordered triple (ρ, θ, ϕ).

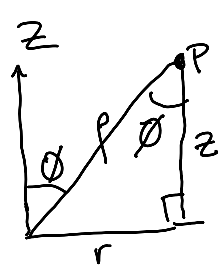
1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the angle between the positive x-axis and the projection of $\overrightarrow{OP}$ in the xy-plane (same θ as in cylindrical coordinates).
3. ϕ is the angle between the positive z-axis and $\overrightarrow{OP}$ $(0 \leq \phi \leq \pi)$

O = origin

Note: Spherical coordinates are especially useful for surfaces that are symmetric about a point, or center.

$\rho^2 = x^2 + y^2 + z^2$





Notice: $r = \rho\sin\phi$

$z = \rho\cos\phi$

$\cos\phi = \dfrac{z}{\rho}$

$z = \rho\cos\phi$

$\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$x = r\cos\theta = \underbrace{\rho\sin\phi}_{r}\cos\theta$

$y = r\sin\theta = \underbrace{\rho\sin\phi}_{r}\sin\theta$

Converting between rectangular and spherical coordinate systems:

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$

$x^2 + y^2 + z^2 = \rho^2 \qquad \tan\theta = \dfrac{y}{x} \qquad \phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$

Converting between cylindrical and spherical coordinate systems $r \geq 0$:

$r^2 = \rho^2 \sin^2\phi \qquad \theta = \theta \qquad z = \rho\cos\phi$

$\hookrightarrow r = \rho\sin\phi$

$\rho = \sqrt{r^2+z^2} \qquad \theta = \theta \qquad \phi = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2+z^2}}\right)$

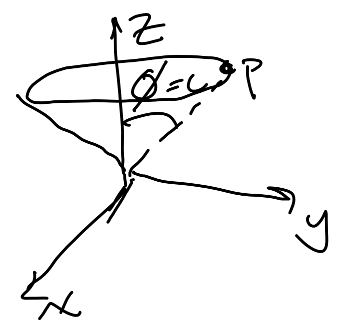
Example 8: Convert the equation $\rho = c$ (c a constant) in spherical coordinates into an equation in rectangular coordinates.

A sphere of radius c

$$x^2+y^2+z^2=\rho^2$$

$$\boxed{x^2+y^2+z^2=c^2}$$

Example 9: Convert the equation $\phi = c$, $0 < c < \dfrac{\pi}{2}$, in spherical coordinates into an equation in rectangular coordinates.



Cone

$$\cos\phi = \frac{z}{\rho}$$

$$\cos c = \frac{z}{\rho}$$

$$\cos c = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$(x^2+y^2+z^2)\cos^2 c = z^2$$

$$x^2+y^2+z^2 = \frac{z^2}{\cos^2 c} \Rightarrow x^2+y^2 = z^2\left(\frac{1}{\cos^2 c} - 1\right)$$

$$x^2+y^2 = z^2\left(\frac{1-\cos^2 c}{\cos^2 c}\right)$$

next page

$x^2+y^2 = z^2\left(\frac{\sin^2 c}{\cos^2 c}\right)$

$x^2+y^2 = z^2\tan^2 c \rightarrow x^2+y^2-\frac{z^2}{\cot^2 c}=0$

Trace xz: $x^2-\frac{z^2}{\cot^2 c}=0 \Rightarrow x^2=\frac{z^2}{\cot^2 c}$

$\frac{x^2}{z^2}=\frac{1}{\cot^2 c}$

$\frac{x^2}{z^2}=\tan^2 c$

Example 10: Convert the equation $x^2+y^2-3z^2=0$ to spherical coordinates from rectangular coordinates.

$\boxed{\text{cone}}$

$x=r\cos\theta$ $\quad$ $y=r\sin\theta$

$x=\underbrace{\rho\sin\phi}_{r}\cos\theta$ $\quad$ $y=\underbrace{\rho\sin\phi}_{r}\sin\theta$

$x^2+y^2-3z^2=0$

$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta-3(\rho\cos\phi)^2=0$

$\rho^2\sin^2\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}-3\rho^2\cos^2\phi=0$

$\rho^2\sin^2\phi-3\rho^2\cos^2\phi=0$

$\rho^2\sin^2\phi=3\rho^2\cos^2\phi$

either $\rho=0$ or

$\sin^2\phi=3\cos^2\phi$

$\frac{\sin^2\phi}{\cos^2\phi}=3$

$\tan^2\phi=3$

$\tan\phi=\sqrt{3}$

$\phi=\frac{\pi}{3}, \frac{2\pi}{3}$

Eqn of surface

$\boxed{\phi=\frac{\pi}{3} \text{ or } \phi=\frac{2\pi}{3}}$

Example 11: Describe the surface with equation $\phi=\frac{\pi}{2}$ in spherical coordinates.

xy plane

Example 12: Describe the surface with equation $\phi=\frac{\pi}{4}$ in spherical coordinates.

top half of cone

Example 13: Convert the point that is represented by $(1,2,3)$ in rectangular coordinates to cylindrical and spherical coordinates.

$r^2=x^2+y^2=1^2+2^2=5$

$r=\sqrt{5}$

$\rho^2=x^2+y^2+z^2=1^2+2^2+3^2=14$

$\rho=\sqrt{14}$

$\tan\theta=\frac{y}{x}=\frac{2}{1}$

$\theta=\tan^{-1}(2)$

$z=\rho\cos\phi$

$\cos\phi=\frac{z}{\rho}=\frac{3}{\sqrt{14}}$

$\phi=\cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$

Cylindrical: $\boxed{(r,\theta,z)=(\sqrt{5}, \tan^{-1}(2), 3)}$

Spherical: $\boxed{(\rho,\theta,\phi)=\left(\sqrt{14}, \tan^{-1}(2), \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)\right)}$