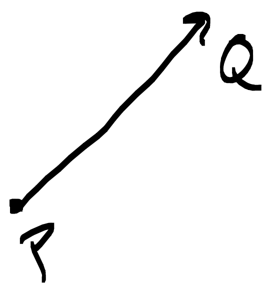
11.1: Vectors in the Plane

Note Title 7/9/2015

Vector: A directed line segment from an initial point to a terminal point.



Q: terminal point

P: initial point.

Directed line segments (vectors) with the same length and direction are considered equal.

Notation for vectors: Bold-faced type, or with arrows on top.

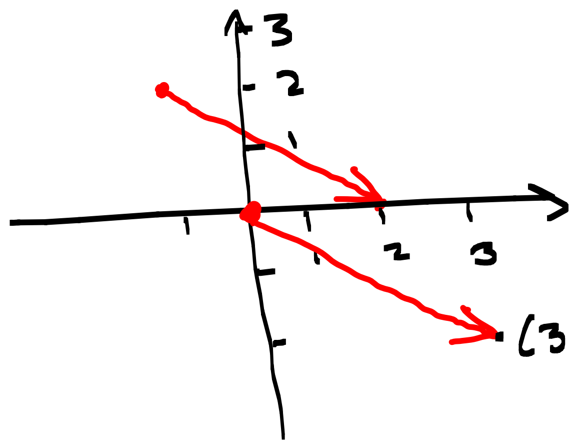
Example: $\overrightarrow{PQ}$ $\quad \vec{u}$ $\quad \vec{v}$

Component form for a vector:

The component form for a vector $\vec{v}$ is given by $\langle v_1, v_2 \rangle$, where v_1 is the horizontal distance from the initial pt to the terminal point, and v_2 is the vertical distance.

Note: The components v_1 and v_2 are scalars (numbers).

Ex: Suppose $\vec{v} = \langle 3, -2 \rangle$



2 copies of the same vector $\vec{v}$.

Note: the vector from initial point $P(p_1, p_2)$ to terminal point $Q(q_1, q_2)$ is $\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle$

Zero vector:

$$\vec{0} = \langle 0, 0 \rangle$$

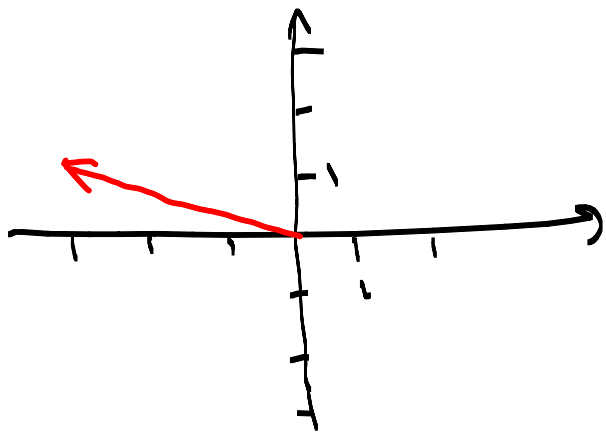
(length is 0; terminal and initial pts are the same.)

Ex: Suppose $P = (-2,3)$ and $Q = (1,2)$.

Find $\overrightarrow{QP}$.

$$\overrightarrow{QP} = \langle -2-1, 3-2 \rangle$$
$$= \langle -3, 1 \rangle$$

(it could have any initial point)



Magnitude (Norm) of a Vector
(length of a vector)

For $\vec{v} = \langle v_1, v_2 \rangle$:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Note: In 1 dimension (real number line), the magnitude is the absolute value.

$$|-5| = 5$$

Unit vector: Any vector with magnitude 1.

i.e. $\vec{v}$ is a unit vector iff $\|\vec{v}\| = 1$.

Shorthand: iff means if and only if. Same as $\Longleftrightarrow$

Above example:

$$\| \langle -3, 1 \rangle \| = \sqrt{(-3)^2 + 1^2} = \sqrt{9+1} = \boxed{\sqrt{10}}$$

Note: Magnitude is a scalar.

Vector operations:

Suppose $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$.

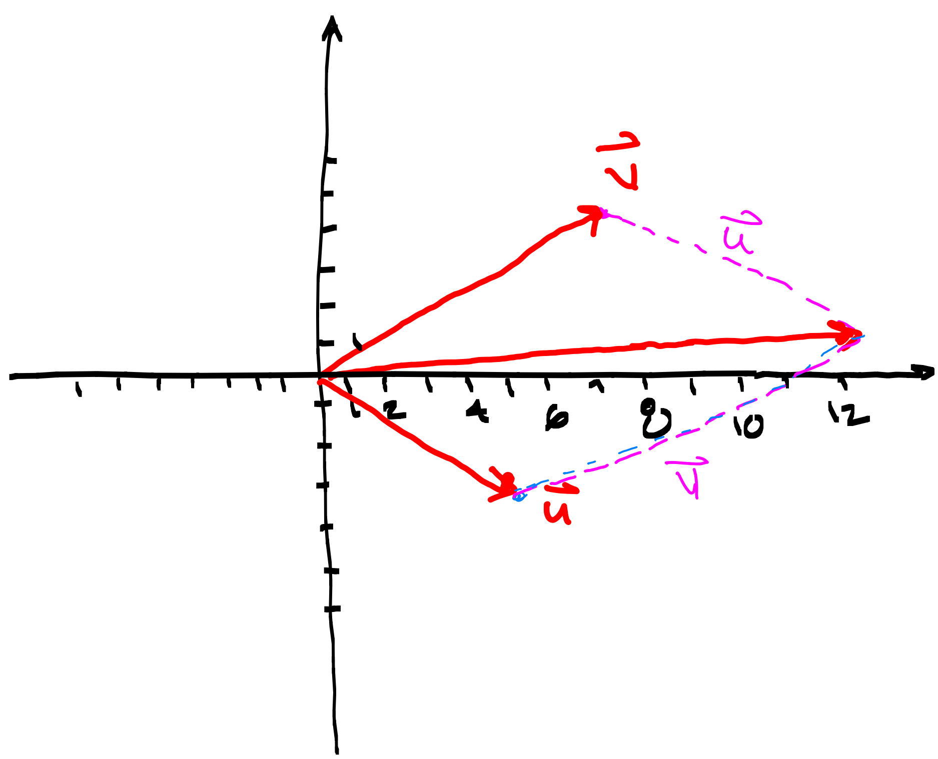
$$\vec{u} \pm \vec{v} = \langle u_1 \pm v_1 , u_2 \pm v_2 \rangle$$

$$c\vec{u} = c\langle u_1, u_2 \rangle = \langle cu_1, cu_2 \rangle \quad (c \text{ is a scalar})$$

Example: Suppose $\vec{u} = \langle 5, -3 \rangle$ and $\vec{v} = \langle 7, 4 \rangle$.

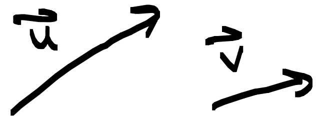
$$\vec{u} + \vec{v} = \langle 5+7, -3+4 \rangle = \langle 12, 1 \rangle$$

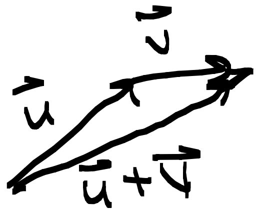
$$\vec{u} - \vec{v} = \langle 5-7, -3-4 \rangle = \langle -2, -7 \rangle$$



Adding vectors geometrically

$\vec{u}+\vec{v}$ is called the resultant vector of $\vec{u}$ and $\vec{v}$





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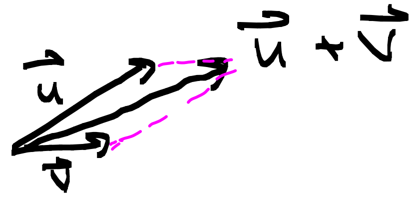
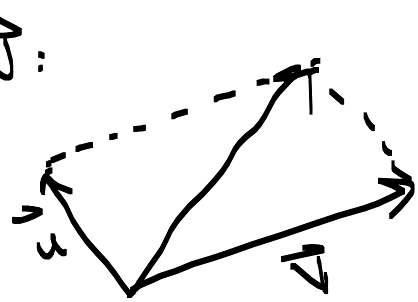


Ex:

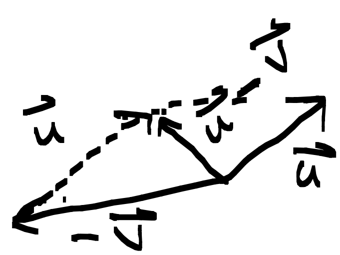
Draw $\vec{u}+\vec{v}$:



Subtraction:

$\vec{u}-\vec{v} \equiv \vec{u}+(-\vec{v})$

Notation: $\equiv$ means "is defined as"



Note: $\langle 3,-4\rangle - \langle -5,2\rangle$
$= \langle 8,-6\rangle$

$\langle 3,-4\rangle - 12$ is not defined.

$12\langle 3,-4\rangle = \langle 36,-48\rangle$

Length of a scalar multiple

Let $\vec{v}$ be a vector and c a scalar.

Then, $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

Previous example:

$$\|12\langle 3,-4\rangle\| = |12|\,\|\langle 3,-4\rangle\|$$
$$= 12\sqrt{9+16}$$
$$= 12(5) = 60$$

Unit Vectors

The unit vector in the direction of $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

(To get the unit vector, divide by the magnitude).
This is called normalizing

Ex: Suppose $\vec{v} = \langle 2, -3 \rangle$.

Find the unit vector in the ⓐ same direction and ⓑ opposite direction of $\vec{v}$.

ⓐ <u>same direction.</u>

Find magnitude: $\|\vec{v}\| = \|\langle 2,-3 \rangle\| = \sqrt{4+9}$
$= \sqrt{13}$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 2,-3 \rangle}{\sqrt{13}} = \frac{1}{\sqrt{13}} \langle 2, -3 \rangle$$

$$= \boxed{\left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle}$$

ⓑ <u>opposite direction:</u>

$$\vec{w} = \left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

<u>Example:</u> Find the vector $\vec{v}$ with magnitude 2, and in the same direction as $\vec{u} = \langle \sqrt{3}, 3 \rangle$.

Find a unit vector in same direction as $\vec{u}$:

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{12}} \langle \sqrt{3}, 3 \rangle = \left\langle \frac{\sqrt{3}}{\sqrt{12}}, \frac{3}{\sqrt{12}} \right\rangle$$

$$= \left\langle \sqrt{\frac{1}{4}}, \frac{3}{2\sqrt{3}} \right\rangle = \left\langle \frac{1}{2}, \frac{3\sqrt{3}}{6} \right\rangle = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

Then multiply by $\|\vec{v}\| = 2$:

$$\vec{v} = 2\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \boxed{\langle 1, \sqrt{3} \rangle}$$

Another approach: All unit vectors have length 1, so

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\frac{\langle \sqrt{3}, 3 \rangle}{\sqrt{12}} = \frac{\vec{v}}{2}$$

multiply both sides by 2:

$$\vec{v} = \frac{2\langle \sqrt{3}, 3 \rangle}{\sqrt{12}} = \frac{2}{2\sqrt{3}} \langle \sqrt{3}, 3 \rangle$$

$$= \frac{1}{\sqrt{3}} \langle \sqrt{3}, 3 \rangle = \left\langle \frac{\sqrt{3}}{3}, \frac{3}{\sqrt{3}} \right\rangle$$

$$= \boxed{\langle 1, \sqrt{3} \rangle} \checkmark$$

Standard Unit Vectors

$\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.

Sometimes written as $\hat{i}$ and $\hat{j}$

$$\langle v_1, v_2 \rangle = v_1 \vec{i} + v_2 \vec{j}$$

Example: $\langle 5, -2 \rangle = 5\vec{i} - 2\vec{j}$

Note: $\langle 5\vec{i} - 2\vec{j} \rangle$ is total nonsense! (don't write this!)

So, every vector can be written 2 ways:

1) Component form: $\vec{v} = \langle a, b \rangle$

2) As a linear combination of standard unit vectors. $\vec{v} = a\vec{i} + b\vec{j}$

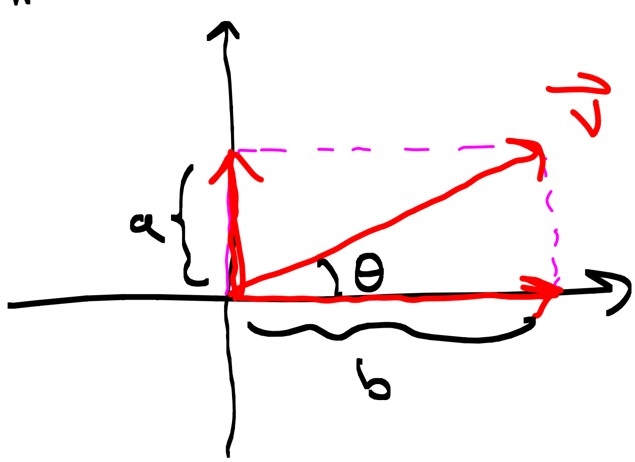
Breaking a vector into horizontal and vertical components

A vector $\vec{v}$ that makes a positive angle θ with the x-axis can be written as

$$\vec{v} = \|\vec{v}\| \cos\theta \, \vec{i} + \|\vec{v}\| \sin\theta \, \vec{j}$$

$\|\vec{v}\| \cos\theta \, \vec{i}$: horizontal component

$\|\vec{v}\| \sin\theta \, \vec{j}$: vertical component



$$\cos\theta = \frac{b}{\|\vec{v}\|}$$

$$\|\vec{v}\| \cos\theta = b$$

$$\sin\theta = \frac{a}{\|\vec{v}\|}$$

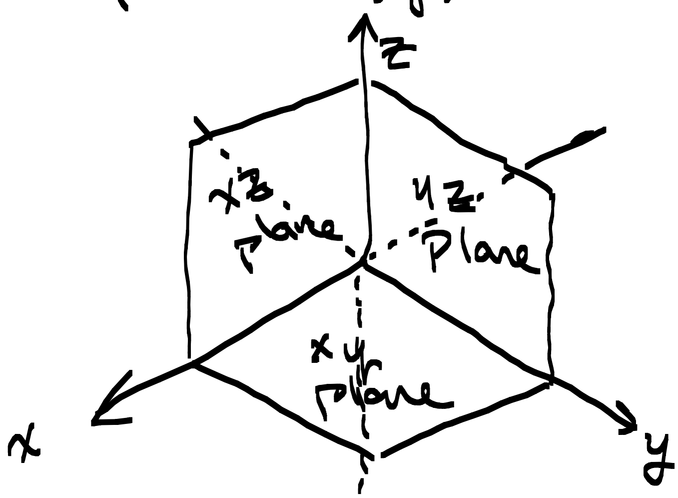
$$\|\vec{v}\| \sin\theta = a$$

11.2: Space Coordinates and Vectors in Space

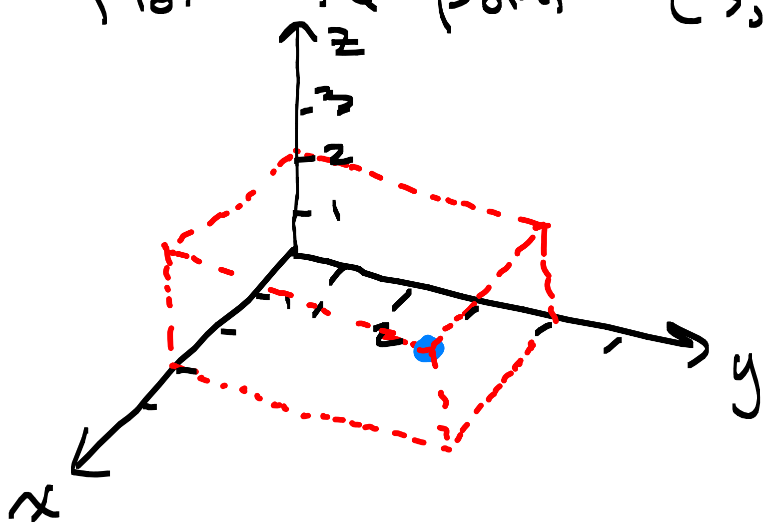
Here, "Space" = $\mathbb{R}^3$ $\Rightarrow$ 3 dimensions

3-dimensional coordinate system:

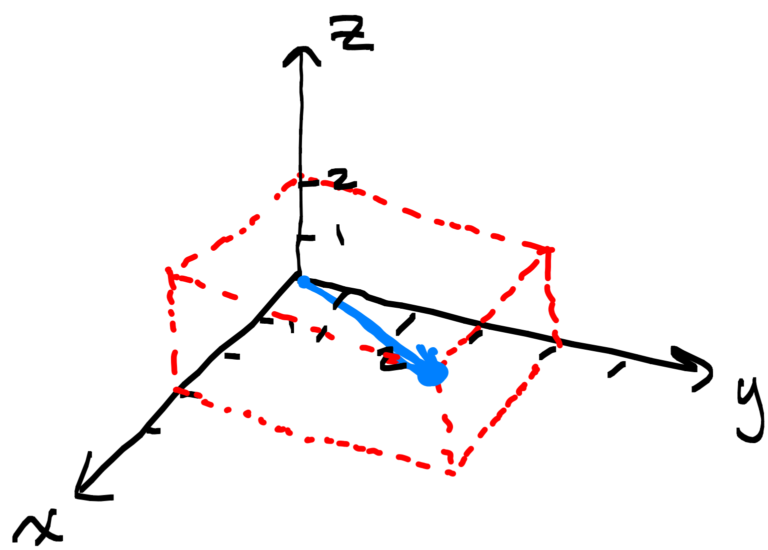
Ordered triples: (x, y, z) to indicate points.



Ex: Plot the point $(3, 4, 2)$.



Use the above picture to draw the vector $\langle 3, 4, 2 \rangle$.



Distance Formula:

The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a sphere:

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2,$$

where r is the radius, and (x_1, y_1, z_1) is the center.

Midpoint Formula:

The midpoint of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Example: Find the center and radius of the sphere with equation

$$x^2+y^2+z^2+9x-2y+10z+19=0$$

$$x^2+9x+\underline{\tfrac{81}{4}}+y^2-2y+\underline{1}+z^2+10z+\underline{25}$$

$$=-19+\underline{\tfrac{81}{4}+1+25}$$

$$\left(x+\frac{9}{2}\right)^2+(y-1)^2+(z+5)^2=\frac{109}{4}$$

Center: $\left(-\frac{9}{2}, 1, -5\right)$

Radius: $\sqrt{\frac{109}{4}} = \frac{\sqrt{109}}{2}$

x: $\left(\frac{9}{2}\right)^2 = \frac{81}{4}$

y: $\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

z: $\left(\frac{10}{2}\right)^2 = 5^2 = 25$

Standard Unit Vectors in $\mathbb{R}^3$

$\vec{i} = \hat{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \hat{j} = \langle 0, 1, 0 \rangle$

$\vec{k} = \hat{k} = \langle 0, 0, 1 \rangle$

If $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the unit vector is

$$\frac{1}{\|\vec{v}\|} \langle v_1, v_2, v_3 \rangle.$$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Parallel Vectors:

<u>Definition</u>: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if there is a scalar c such that $\vec{u} = c\vec{v}$.

<u>Ex</u>: Are these two vectors parallel?

$$\vec{u} = \langle 2, -3, 4 \rangle, \quad \vec{v} = \langle -6, 9, -12 \rangle$$

$\boxed{\text{Yes, because } -3\vec{u} = \vec{v}}$

Triangle Inequality (applies to both $\mathbb{R}^2$ and $\mathbb{R}^3$)

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

Ex: Use vectors to determine if these points are collinear.

$A(4,-2,7)$
$B(-2,0,3)$
$C(7,-3,9)$

Find vectors:

$$\overrightarrow{AB} = \langle -2-4, 0-(-2), 3-7\rangle = \langle -6, 2, -4\rangle$$

$$\overrightarrow{BC} = \langle 7-(-2), -3-0, 9-3\rangle = \langle 9, -3, 6\rangle$$

$$\overrightarrow{AC} = \langle 7-4, -3-(-2), 9-7\rangle = \langle 3, -1, 2\rangle$$

$$\|\overrightarrow{AB}\| = \sqrt{36+4+16} = \sqrt{56} = \sqrt{4\cdot 14} = 2\sqrt{14}$$

$$\|\overrightarrow{BC}\| = \sqrt{81+9+36} = \sqrt{126} = \sqrt{9\cdot 14} = 3\sqrt{14}$$

$$\|\overrightarrow{AC}\| = \sqrt{9+1+4} = \sqrt{14}$$

$$\left.\begin{array}{l} \|\overrightarrow{AC}\| + \|\overrightarrow{AB}\| = \|\overrightarrow{BC}\| \\ \sqrt{14} + 2\sqrt{14} = 3\sqrt{14} \end{array}\right\}$$ So yes, they are collinear.

11.3: The Dot Product (of 2 vectors)

Definition of the Dot Product

For $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, the dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

For $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example:

$$\langle 4, -2, 3 \rangle \cdot \langle 2, 5, -4 \rangle$$

$$= 4(2) + (-2)(5) + 3(-4)$$

$$= 8 - 10 - 12 = \boxed{-14}$$

Properties of the dot product

1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

<u>Proof</u>: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{w} = \langle w_1, w_2, w_3 \rangle$

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Previous proof cont'd:

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \langle u_1, u_2, u_3\rangle \cdot (\langle v_1, v_2, v_3\rangle + \langle w_1, w_2, w_3\rangle)$$

$$= \langle u_1, u_2, u_3\rangle \cdot \langle v_1+w_1, v_2+w_2, v_3+w_3\rangle$$

$$= u_1(v_1+w_1) + u_2(v_2+w_2) + u_3(v_3+w_3)$$

$$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + u_3v_3 + u_3w_3$$

$$= u_1v_1 + u_2v_2 + u_3v_3 + u_1w_1 + u_2w_2 + u_3w_3$$

$$= \langle u_1, u_2, u_3\rangle \cdot \langle v_1, v_2, v_3\rangle + \langle u_1, u_2, u_3\rangle \cdot \langle w_1, w_2, w_3\rangle$$

$$= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \quad \blacksquare$$

<u>More properties of dot product:</u>

3) $c(\vec{u}\cdot\vec{v}) = c\vec{u}\cdot\vec{v} = \vec{u}\cdot c\vec{v}$

4) $\vec{0}\cdot\vec{v} = 0$

* 5) $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

<u>Proof</u>: For $v = \langle v_1, v_2, v_3\rangle$, then

$$\vec{v}\cdot\vec{v} = \langle v_1, v_2, v_3\rangle \cdot \langle v_1, v_2, v_3\rangle$$

$$= v_1v_1 + v_2v_2 + v_3v_3$$

$$= v_1^2 + v_2^2 + v_3^2 = \|\vec{v}\|^2$$

Angle between 2 vectors

For vectors $\vec{u}$ and $\vec{v}$ with angle θ between them ($0 \le \theta \le 180°$), then

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}}$$

So, the angle between $\vec{u}$ and $\vec{v}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

Ex.: Find the angle between the vectors. $\vec{u} = \langle 3,-5,7\rangle$ and $\vec{v} = \langle 1,2,3\rangle$.

$$\|\vec{u}\| = \sqrt{9+25+49} = \sqrt{83}$$

$$\|\vec{v}\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\vec{u}\cdot\vec{v} = 3(1)-5(2)+7(3) = 3-10+21 = 14$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\cos\theta = \frac{14}{\sqrt{83}\sqrt{14}} \approx 0.4107$$

$$\theta \approx \cos^{-1}(0.4107) \approx \boxed{65.75°}$$

Useful Result of $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$:

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

(Sometimes called the alternative dot product)

Orthogonal Vectors

(meet at 90° angle)

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if
$\vec{u}\cdot\vec{v} = 0$. (because $\cos\theta) = 0$

Vectors will be parallel if
$\cos\theta = \pm 1$.