11.3 (Dot Product) Cont'd:

Note Title 7/13/2015

Recall: $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

If $\vec{u}$ and $\vec{v}$ are orthogonal, $\theta = 90°$

and thus $\cos\theta = 0$.

Orthogonal Vectors:

$\vec{u}$ and $\vec{v}$ are orthogonal iff $\vec{u}\cdot\vec{v} = 0$.

Ex: Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal, ~~perpendicular~~ parallel, or neither.

$\vec{u} = \langle -4, 6, 10\rangle$, $\vec{v} = \langle 6, -9, 15\rangle$

x-components: $-4t = 6$

$t = \frac{6}{-4} = -\frac{3}{2}$

Try other components with $t = -\frac{3}{2}$:

$6\left(-\frac{3}{2}\right) = -9$ ✓

$10\left(-\frac{3}{2}\right) = -15$ No, not parallel.

Note: $\langle -4, 6, 10\rangle$ and $\langle 6, -9, -15\rangle$ are parallel.

Are they orthogonal?

$\vec{u}\cdot\vec{v} = \langle -4, 6, 10\rangle \cdot \langle 6, -9, 15\rangle$

$= -24 - 54 + 150 \neq 0$. so not orthogonal.

$\boxed{\text{Neither}}$

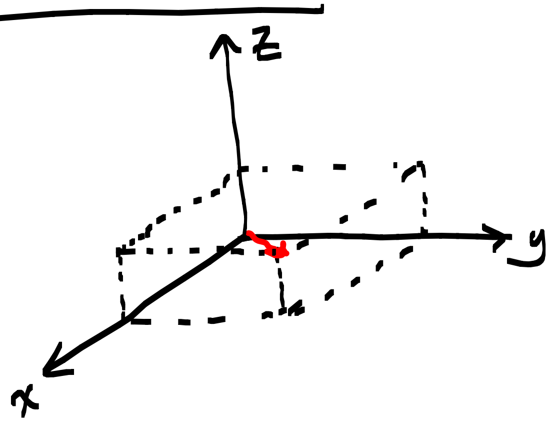
Ex: Are $\vec{u}$ & $\vec{v}$ orthogonal, parallel, or neither?

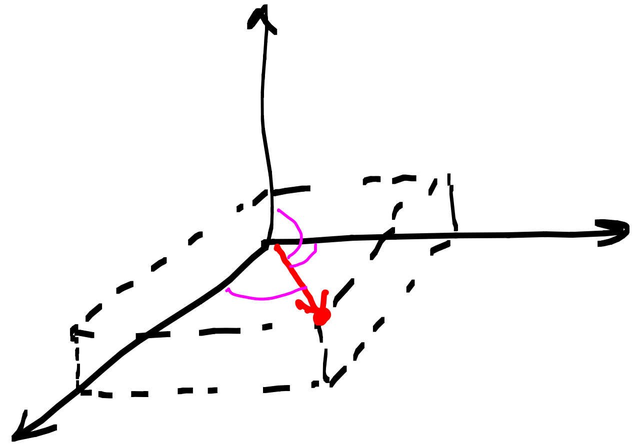
$\vec{u} = \langle 2, -5, 3 \rangle$, $\vec{v} = \langle 6, 3, 1 \rangle$.

$\vec{u} \cdot \vec{v} = 12 - 15 + 3 = 0$.

$\boxed{\text{They are orthogonal.}}$

<u>Direction Cosines</u>:





Let α = angle between $\vec{v}$ and the x-axis

β = angle between $\vec{v}$ and the y-axis

gamma $\longrightarrow$ γ = angle between $\vec{v}$ and the z-axis.

Suppose $\vec{v} = \langle v_1, v_2, v_3 \rangle$

<u>Direction cosines</u> are:

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_2}{\|\vec{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\vec{v}\|}$$

So, the <u>direction angles</u> α, β, γ are given by:

$$\alpha = \cos^{-1}\left(\frac{v_1}{\|\vec{v}\|}\right), \quad \beta = \cos^{-1}\left(\frac{v_2}{\|\vec{v}\|}\right), \quad \gamma = \cos^{-1}\left(\frac{v_3}{\|\vec{v}\|}\right)$$

<u>Note</u>: The unit vector $\dfrac{\vec{v}}{\|\vec{v}\|} = \left\langle \dfrac{v_1}{\|\vec{v}\|}, \dfrac{v_2}{\|\vec{v}\|}, \dfrac{v_3}{\|\vec{v}\|} \right\rangle = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

Ex: Find the direction angles (in degrees) for $\vec{u} = \langle -2, 6, 1 \rangle$.

$$\|\vec{u}\| = \sqrt{4+36+1} = \sqrt{41}$$

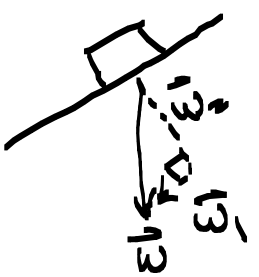
$$\cos\alpha = \frac{-2}{\sqrt{41}} \Rightarrow \alpha = \cos^{-1}\left(\frac{-2}{\sqrt{41}}\right) = 108.2^\circ$$

$$\cos\beta = \frac{6}{\sqrt{41}} \Rightarrow \beta = 20.44^\circ$$

$$\cos\gamma = \frac{1}{\sqrt{41}} \Rightarrow \gamma = 81.02^\circ$$

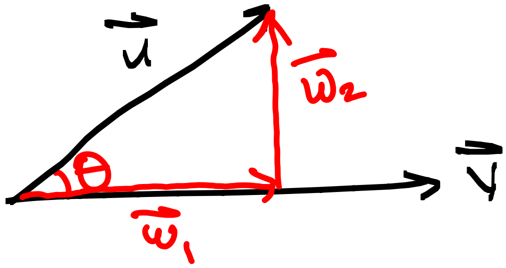
Note: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, because $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector.

Projections and Vector Components:



$$\vec{F} = \vec{w}_1 + \vec{w}_2$$

$$\vec{v} = \vec{v}_1 + \vec{v}_2$$



Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let $\vec{u} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1$ is parallel to $\vec{v}$ and $\vec{w}_2$ is orthogonal to $\vec{v}$. Then,

1) $\vec{w}_1$ is called the projection of $\vec{u}$ onto $\vec{v}$ and is denoted $\vec{w}_1 = \text{proj}_{\vec{v}}\vec{u}$.

2) $\vec{w}_2 = \vec{u} - \vec{w}_1$.

Theorem:

$$\boxed{\vec{w}_1 = \text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}}$$

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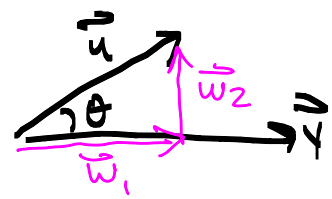


Why?

$$\sin\theta = \frac{\|\vec{w}_2\|}{\|\vec{u}\|} \qquad \cos\theta = \frac{\|\vec{w}_1\|}{\|\vec{u}\|}$$

$$\|\vec{w}_2\| = \|\vec{u}\|\sin\theta \qquad \|\vec{w}_1\| = \|\vec{u}\|\cos\theta$$

$$= \|\vec{u}\|\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

Multiply this by a unit vector to get $\vec{w}_1$.

$$\vec{w}_1 = \|\vec{w}_1\|\frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

Ex: Suppose $\vec{u} = \langle 3, -5, 2 \rangle$ and $\vec{v} = \langle 2, 3, -4 \rangle$.

Find $\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u}$ and $\vec{w}_2$.

$$\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$= \frac{6 - 15 - 8}{\left(\sqrt{4+9+16}\right)^2} \langle 2, 3, -4 \rangle$$

$$= \frac{-17}{29} \langle 2, 3, -4 \rangle$$

$$= \boxed{\left\langle \frac{-34}{29}, \frac{-51}{29}, \frac{68}{29} \right\rangle}$$

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = \langle 3, -5, 2 \rangle - \left\langle \frac{-34}{29}, \frac{-51}{29}, \frac{68}{29} \right\rangle$$

$$= \boxed{\left\langle \frac{121}{29}, \frac{-94}{29}, \frac{-10}{29} \right\rangle}$$

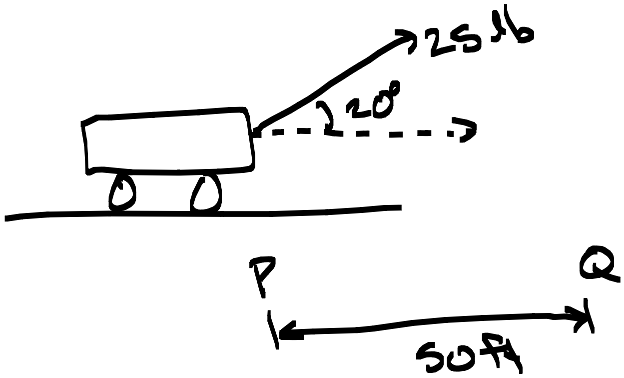
Work:

The work done by force $\vec{F}$ in moving an object from P to Q is

$$W = \vec{F} \cdot \vec{PQ}, \text{ or}$$

$$W = \|\text{proj}_{\vec{PQ}} \vec{F}\| \, \|\vec{PQ}\|$$

Example: A force of 25 lbs is applied to a wagon at an angle of 20° above the horizontal. Find the work done in pulling the wagon 50 ft.



$$\vec{F} = \langle 25\text{ lb}\cos 20^\circ, 25\text{ lb}\sin 20^\circ \rangle$$

$$\overrightarrow{PQ} = \langle 50\text{ ft}, 0 \rangle$$

$$W = \vec{F}\cdot\overrightarrow{PQ} = (25\text{ lb}\cos 20^\circ)(50\text{ ft}) + 0$$

$$= \boxed{1174.6\text{ ft}\cdot\text{lb}}$$