11.4: The Cross Product

The *cross product* (or *vector product*) of two vectors in $\mathbb{R}^3$ (3-dimensional space) yields a vector that is orthogonal to both of the vectors that produced it.

> Definition: The Cross Product
>
> Suppose that $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. The cross product of **u** and **v** is the vector
>
> $$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle \\ &= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle. \end{aligned}$$

Note: The cross product is not defined for two-dimensional vectors.

The determinant:

The determinant is a concept from linear algebra. The determinant is a characteristic of square matrices, but it can help us calculate the cross product of two vectors.

The determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant of a 3×3 matrix is

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ek - fh) - b(dk - fg) + c(dh - eg). \end{aligned}$$

Example 1: Find the determinant of $\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$.

Determinant of $\begin{bmatrix} 2 & 5 \\ 3 & 9 \end{bmatrix}$ is $\begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} = 2(9) - 5(3) = 18 - 15 = 3$

Example 2: Find the determinant of $\begin{bmatrix} 2 & -7 \\ -1 & -9 \end{bmatrix}$.

$$\begin{vmatrix} 2 & -7 \\ -1 & -9 \end{vmatrix} = 2(-9) - (-7)(-1) = -18 - 7 = \boxed{-25}$$

Example 3: Find the determinant of $\begin{bmatrix} 4 & 2 & -3 \\ 7 & 5 & -8 \\ -2 & 0 & 1 \end{bmatrix}$.

$$\begin{vmatrix} 4 & 2 & -3 \\ 7 & 5 & -8 \\ -2 & 0 & 1 \end{vmatrix} = 4\begin{vmatrix} 5 & -8 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 7 & -8 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & 5 \\ -2 & 0 \end{vmatrix}$$

$$= 4(5-0) - 2(7-16) - 3(0-(-10))$$

$$= 4(5) - 2(-9) - 3(10) = 20 + 18 - 30 = \boxed{8}$$

The determinant approach to calculating the cross product.

Put the standard unit vectors **i**, **j**, and **k** in Row 1, the first vector in Row 2, and the second vector in Row 3.

The cross product of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$
$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Note: This is technically not a determinant, because the first row (containing **i**, **j**, and **k**) contains vectors, not scalars.

Example 4: Suppose $\mathbf{u} = \langle 3, 1, -2\rangle$ and $\mathbf{v} = \langle -4, 2, 6\rangle$. Calculate $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$. Show that the cross product is orthogonal to both of the original vectors.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ -4 & 2 & 6 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & -2 \\ 2 & 6 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & -2 \\ -4 & 6 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ -4 & 2 \end{vmatrix}$$

$$= \hat{i}(6-(-4)) - \hat{j}(18-8) + \hat{k}(6-(-4))$$

$$= 10\hat{i} - 10\hat{j} + 10\hat{k} = \boxed{\langle 10, -10, 10\rangle}$$

show it's orthogonal to $\vec{u}$: $\langle 10,-10,10\rangle \cdot \langle 3,1,-2\rangle = 30 - 10 - 20 = 0$ ✓ok

show it's $\perp$ to $\vec{v}$: $\boxed{\langle 10,-10,10\rangle \cdot \langle -4,2,6\rangle = -40 - 20 + 60 = 0} $ ✓

Example 5: Suppose $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Calculate $\mathbf{u}\times\mathbf{v}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 6 & 0 \\ -2 & 1 & 1 \end{vmatrix} = \langle 6-0, -(1-0), 1-(-12)\rangle$$

$$= \boxed{\langle 6, -1, 13\rangle}$$

Properties of the cross product:

Algebraic properties of the cross product:

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

1. $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$
2. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times\mathbf{w})$
3. $c(\mathbf{u}\times\mathbf{v}) = c\mathbf{u}\times\mathbf{v} = \mathbf{u}\times c\mathbf{v}$
4. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \vec{0}$
5. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$

Also: $\vec{u}\times\vec{u} = \vec{0}$

Geometric properties of the cross product:

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^3$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then,

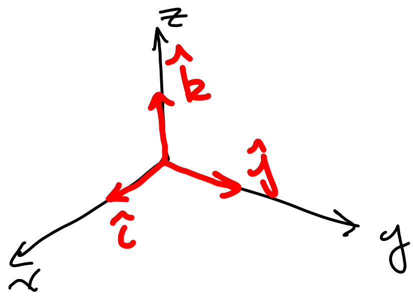
6. $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
7. $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$
8. $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
9. $\|\mathbf{u}\times\mathbf{v}\|$ is the area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

   Note: This means that $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$ is the area of a triangle having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

The right-hand rule:

The cross product follows what is known as the *right-hand rule*. This means that if you curl the fingers of your right hand from vector $\mathbf{u}$ to vector $\mathbf{v}$, your thumb will point in the direction of $\mathbf{u}\times\mathbf{v}$.

Note: This means that $\mathbf{k} = \mathbf{i}\times\mathbf{j}$.



Example 6: Suppose $\mathbf{u} = \langle 2, -1, 3 \rangle$ and $\mathbf{v} = \langle -4, 2, -6 \rangle$. Calculate $\mathbf{u} \times \mathbf{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ -4 & 2 & -6 \end{vmatrix} = \langle 6-6, -12-(-12), 4-4 \rangle$$

$$= \langle 0, 0, 0 \rangle$$ They are parallel.

Note: $-2\vec{u} = \vec{v}$.

Example 7: Find the area of the triangle with vertices $A(2,-3,4)$, $B(0,1,2)$, and $C(-1,2,0)$.

$$\overrightarrow{AB} = \langle 0-2, 1-(-3), 2-4 \rangle = \langle -2, 4, -2 \rangle$$

$$\overrightarrow{AC} = \langle -1-2, 2-(-3), 0-4 \rangle = \langle -3, 5, -4 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{vmatrix} = \langle -16+10, 6-8, -10-(-12) \rangle$$

$$= \langle -6, -2, 2 \rangle$$

Area of triangle $= \frac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{36+4+4} = \frac{1}{2}\sqrt{44} = \frac{2\sqrt{11}}{2}$

$= \boxed{\sqrt{11}}$

Example 8: Suppose the following points are the vertices of a quadrilateral. Determine whether the quadrilateral is a parallelogram. Find the area.

$B(9,-1,-2)$

$A(1,1,3)$, $B\cancel{(9,-1,2)}$, $C(11,2,-9)$, and $D(3,4,-4)$

$\overrightarrow{AB} = \langle 9-1, -1-1, -2-3 \rangle = \langle 8, -2, -5 \rangle$ ✓

$\overrightarrow{AC} = \langle 10, 1, -12 \rangle$

$\overrightarrow{AD} = \langle 2, 3, -7 \rangle$ ✓✓

$\overrightarrow{BC} = \langle 2, 3, -7 \rangle$ ✓✓ equal to $\overrightarrow{AD}$

$\overrightarrow{BD} = \langle -6, 5, -2 \rangle$

$\overrightarrow{CD} = \langle -8, 2, 5 \rangle$ ✓ parallel to $\overrightarrow{AB}$

Notice: $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\overrightarrow{BC} = \overrightarrow{AD}$, so it's a parallelogram

Find the area:

$$\overrightarrow{AD} \times \overrightarrow{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -7 \\ 8 & -2 & -5 \end{vmatrix} = \langle -15-14, -56-(-10), -4-24 \rangle$$

$$= \langle -29, -46, -28 \rangle$$

Area $= \|\overrightarrow{AD} \times \overrightarrow{AB}\| = \sqrt{29^2 + 46^2 + 28^2} = \boxed{\sqrt{3741}} \approx 61.16$

The triple scalar product:

The dot product of $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ is called the *triple scalar product*.

<u>Theorem (Triple Scalar Product)</u>:

Suppose $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$, $\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$, and $\mathbf{w}=w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k}$. Then $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is the determinant of the matrix that has **u**, **v**, and **w** as Row 1, Row 2, and Row 3, respectively.

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

<u>Theorem (Volume of a Parallelepiped)</u>:

The volume of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is

$$V=|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$$

<u>Example 9:</u> Find the volume of the parallelepiped having adjacent edges $\mathbf{u}=\langle 1,3,1\rangle$, $\mathbf{v}=\langle 0,6,6\rangle$, and $\mathbf{w}=\langle -4,0,-4\rangle$.

$$\vec{u}\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} 1 & 3 & 1 \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix}=1(-24-0)-3(0-(-24))+1(0-(-24))$$

$$=\cancel{-24}-3(24)+\cancel{24}$$

$$=-72$$

$$\text{Volume}=|\vec{u}\cdot(\vec{v}\times\vec{w})|=|-72|=\boxed{72}$$

Using the cross product to find torque:

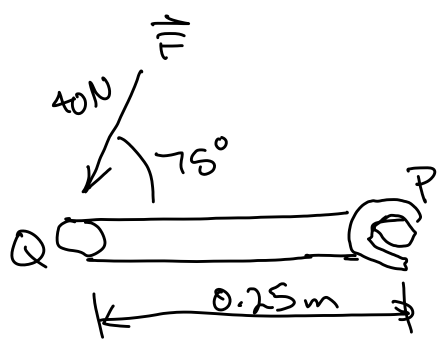
Suppose a force **F** is applied at the point Q. The torque, or moment **M** of the force **F** about a point P, measures the tendency of $\overrightarrow{PQ}$ to rotate counterclockwise about the point P.

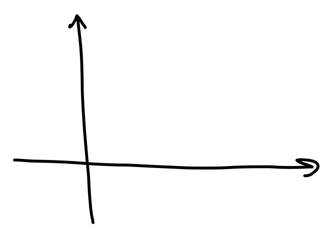
The torque is given by

$$M = \overrightarrow{PQ} \times \mathbf{F} \ .$$

loosened

Example 10: Suppose a bolt is tightened by applying a force of 40 N to a wrench that is 0.25 m long. The force is applied at an angle of $75°$ to the axis of the wrench. Find the magnitude of the torque about the center of the bolt.





$$\overrightarrow{PQ} = \langle -0.25, 0, 0 \rangle$$

$$\vec{F} = \langle -40N\cos 75°, -40N\sin 75°, 0 \rangle$$

$$\overrightarrow{PQ} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -0.25 & 0 & 0 \\ -40N\cos 75° & -40N\sin 75° & 0 \end{vmatrix} = \langle 0, 0, 0.25^{m}(40)N \sin 75° \rangle$$

$$= 10 \sin 75° \ N \cdot m$$

$$\approx 9.66 \ N \cdot m$$

Homework Qs

11.3 #25 Vertices of a triangle are $A(0,0,0)$, $B(1,2,0)$, $C(-2,1,0)$. Classify the triangle as right, acute, or obtuse.

$\overrightarrow{AB} = \langle 1, 2, 0 \rangle$

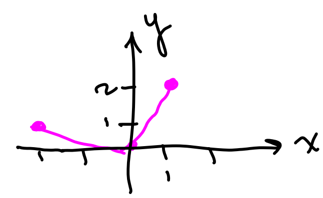
$\overrightarrow{AC} = \langle -2, 1, 0 \rangle$

$\overrightarrow{BC} = \langle -2-1, 1-2, 0-0 \rangle = \langle -3, -1, 0 \rangle$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$$\cos(\measuredangle BAC) = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \frac{\langle 1,2,0\rangle\cdot\langle -2,1,0\rangle}{\sqrt{1+4}\,\sqrt{4+1}} = \frac{-2+2}{5} = \frac{0}{5} = 0$$

so $\measuredangle BAC = 90^\circ$



$\boxed{\text{This is a right triangle.}}$