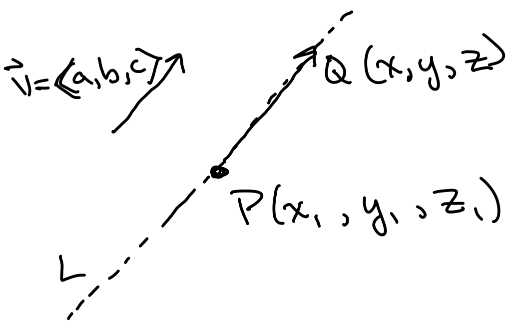
11.5: Lines and Planes in Space

In the plane, a slope and a point are enough to uniquely determine a line. That is not the case for lines in 3-dimensional space ($\mathbb{R}^3$).

In $\mathbb{R}^3$, we need a point and a *direction vector* **v**.



Note: $\overrightarrow{PQ} = t\vec{v}$, where t is a scalar.

$\langle x-x_1, y-y_1, z-z_1 \rangle = t\langle a,b,c \rangle$
$= \langle ta, tb, tc \rangle$

So, $x-x_1 = ta$, $y-y_1 = tb$, $z-z_1 = tc$
$x = x_1+ta$, $y=y_1+tb$, $z=z_1+tc$

Parametric equations of a line in $\mathbb{R}^3$:

Suppose the line L passes through the point $P(x_1, y_1, z_1)$ and is parallel to the vector $\mathbf{v} = \langle a,b,c \rangle$. Then the line can be represented by the following parametric equations:

Vector eqn

$\langle x,y,z \rangle = \langle x_1,y_1,z_1 \rangle + t\langle a,b,c \rangle$

$$\left.\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}\right\}$$

Solve for t:

$t = \dfrac{x-x_1}{a}$

$t = \dfrac{y-y_1}{b}$

$t = \dfrac{z-z_1}{c}$

The scalars *a*, *b*, and *c* are sometimes called the *direction numbers* for the line.

If *a*, *b*, and *c* are all nonzero, the line can also be represented by the symmetric equations

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

Example 1: Find the parametric and symmetric equations for the line that passes through the point $P(4,1,-3)$ and is parallel to $\mathbf{v} = -2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Find two additional points on the line.

$(x_1, y_1, z_1) = (4,1,-3)$, $\vec{v} = \langle a,b,c \rangle = \langle -2,-1,7 \rangle$

$\langle x,y,z \rangle = \langle 4,1,-3 \rangle + t\langle -2,-1,7 \rangle$

parametric eqns for line

$$\boxed{\begin{aligned} x &= 4-2t \\ y &= 1-t \\ z &= -3+7t \end{aligned}}$$

Solve for t: $t = \dfrac{x-4}{-2}$

$t = \dfrac{y-1}{-1}$

$t = \dfrac{z+3}{7}$

$$\boxed{\frac{x-4}{-2} = \frac{y-1}{-1} = \frac{z+3}{7}}$$ symmetric eqns

see next page

To find additional points on the line, choose some t-values.

$t=1 \Rightarrow x = 4-2(1) = 2$, $y = 1-1 = 0$, $z = -3+7(1) = 4$

$t=2 \Rightarrow x = 4-2(2) = 0$, $y = 1-2 = -1$, $z = -3+7(2) = 11$

Two additional pts are $(2,0,4)$ and $(0,-1,11)$

Example 2: Find parametric and symmetric equations for the line that passes through the points $P(-3,0,2)$ and is parallel to $\mathbf{v} = \langle 0,6,3 \rangle$. Find two additional points on the line.

$\langle x,y,z \rangle = \langle -3,0,2 \rangle + t\langle 0,6,3 \rangle$

$$\left.\begin{aligned} x &= -3+0t \\ y &= 0+6t \\ z &= 2+3t \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} x &= -3 \\ y &= 6t \\ z &= 2+3t \end{aligned}}$$ parametric eqns

Solve for t:

$t = \frac{y}{6}$

$t = \frac{z-2}{3}$

symmetric eqns: $\boxed{\frac{y}{6} = \frac{z-2}{3}, \quad x = -3}$

Example 3: Find parametric and symmetric equations for the line that passes through the points $P(2,0,2)$ and $Q(1,4,-3)$.

The direction vector is $\overrightarrow{PQ} = \langle 1-2, 4-0, -3-2 \rangle = \langle -1,4,-5 \rangle$

start with $\langle x,y,z \rangle = \langle 2,0,2 \rangle + t\langle -1,4,-5 \rangle$

or $\langle x,y,z \rangle = \langle 1,4,-3 \rangle + t\langle -1,4,-5 \rangle$ and proceed as before

$$\left.\begin{aligned} x &= 2-t \\ y &= 0+4t \\ z &= 2-5t \end{aligned}\right\} t = \frac{x-2}{-1} = \frac{y-0}{4} = \frac{z-2}{-5}$$

$$\left\{\begin{aligned} x &= 1-1t \\ y &= 4+4t \\ z &= -3-5t \end{aligned}\right.$$

$t = \frac{x-1}{-1} = \frac{y-4}{4} = \frac{z+3}{-5}$

Example 4: Find parametric and symmetric equations for the line that passes through the points $P(-3,5,4)$ and is parallel to the line with symmetric equations $\frac{x-1}{3} = \frac{y+1}{-2} = z-3$.

direction vector: $\langle 3,-2,1 \rangle = \vec{v}$

$$t = \frac{x-1}{3} = \frac{y+1}{-2} = z-3 \Rightarrow \left.\begin{aligned} x &= 1+3t \\ y &= -1-2t \\ z &= t+3 \end{aligned}\right\} \vec{v} = \langle 3,-2,1 \rangle$$

$$\boxed{\begin{aligned} x &= -3+3t \\ y &= 5-2t \\ z &= 4+t \end{aligned}}$$

Equations of planes in $\mathbb{R}^3$:

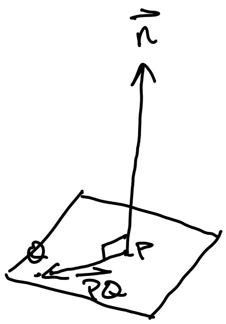
To determine the equation of a line in $\mathbb{R}^2$, we need a point on the line and a slope.
To determine the equation of a line in $\mathbb{R}^3$, we need a point on the line and a direction vector.
To determine the equation of a plane in $\mathbb{R}^3$, we need a point on the plane and a vector that is *normal* to the plane (it forms an angle of $90°$ with any vector in the plane).

<u>Standard Form for the Equation of a Plane</u>:

Suppose a plane contains the point $P(x_1, y_1, z_1)$ and has normal vector $\mathbf{n} = \langle a, b, c\rangle$. Then the *standard form* for the equation of the plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This equation can be rearranged into the form $ax + by + cz + d = 0$, which is called the *general form* for the equation of the plane.



Suppose $P = (x_1, y_1, z_1)$ and $Q(x, y, z)$ are in the plane.

$\vec{n} = \langle a, b, c\rangle$

$\vec{PQ} = \langle x - x_1, y - y_1, z - z_1\rangle$

$\vec{n} \perp \vec{PQ} \Rightarrow \vec{n} \cdot \vec{PQ} = 0 \Rightarrow \langle a, b, c\rangle \cdot \langle x - x_1, y - y_1, z - z_1\rangle = 0$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

<u>Example 5:</u> Find an equation of the plane that includes the points $\underset{P}{(3,-1,2)}$, $\underset{Q}{(2,1,5)}$, and $\underset{R}{(1,-2,-2)}$.

$\vec{PQ} = \langle -1, 2, 3\rangle$

$\vec{PR} = \langle -2, -1, -4\rangle$

To find a normal vector, find $\vec{PQ} \times \vec{PR}$:

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ -2 & -1 & -4 \end{vmatrix} = \langle -8 - (-3), -6 - (+4), 1 - (-4)\rangle$$

$$= \langle -5, -10, 5\rangle$$

$$-5(x - 3) - 10(y - (-1)) + 5(z - 2) = 0$$

Example 6: Suppose a line passes through the point $(-4,5,2)$ and is normal to the plane with equation $-x+2y+z=5$. Find a set of parametric equations for the line. Also, find the point where the line intersects the plane.

To find eqns for line: need a point and a direction vector.
normal vector for plane is $\vec{n} = \langle -1,2,1\rangle$. This is also a direction vector for the line

Vector eqn for a line

$\langle x,y,z\rangle = \langle -4,5,2\rangle + t\langle -1,2,1\rangle$

Parametric eqns for line:

$$\boxed{\begin{aligned} x &= -4-t \\ y &= 5+2t \\ z &= 2+t \end{aligned}}$$

To find the point where the line intersects the plane, substitute parametric representations of x,y,z into eqn for plane.

$-x+2y+z=5$

$-(-4-t)+2(5+2t)+(2+t)=5$

$4+t+10+4t+2+t=5$

$6t+16=5$

$6t=-11$

$t=-\frac{11}{6}$

$x=-4-(-\frac{11}{6})$
$=-4+\frac{11}{6}$
$=-\frac{13}{6}$

$y=5+2(-\frac{11}{6})$
$=5-\frac{11}{3}$
$=\frac{4}{3}$

$z=2-\frac{11}{6}=\frac{1}{6}$

Intersection Point is $\boxed{\left(-\frac{13}{6},\frac{4}{3},\frac{1}{6}\right)}$

Example 7: Find an equation for the plane that passes through the point $(3,2,2)$ and is perpendicular to the line given by $\frac{x-1}{4} = y+2 = \frac{z+3}{-3}$. Write the equation of the plane in general form.

Direction vector for line must be normal vector for plane.

$\vec{n} = \langle 4,1,-3\rangle$

Eqn for plane: $4(x-3)+1(y-2)-3(z-2)=0$

$4x-12+y-2-3z+6=0$

$\boxed{4x+y-3z-8=0}$

Example 8: Find an equation for the plane that passes through the point $(1,2,3)$ and is parallel to the *yz*-plane.

Parallel to yz-plane, so normal vector is $\hat{i} = \langle 1,0,0\rangle$

Eqn of plane is $\boxed{x=1}$

Example 9: Find the point of intersection of the plane with equation $2x+3y=-5$, and the line given by $\frac{x-1}{4}=\frac{y}{2}=\frac{z-3}{6}$.

Write the parametric eqns for the line: $t=\frac{x-1}{4}=\frac{y}{2}=\frac{z-3}{6}$

parametric eqns for line $\begin{cases} x=4t+1 \\ y=2t \\ z=6t+3 \end{cases}$

Put these into $2x+3y=-5$

$$2(4t+1)+3(2t)=-5$$
$$8t+2+6t=-5$$
$$14t=-7$$
$$t=-\frac{1}{2}$$

Put $t=-\frac{1}{2}$ into parametric eqns:

$x=4(-\frac{1}{2})+1=-1$

$y=2(-\frac{1}{2})=-1$

$z=6(-\frac{1}{2})+3=0$

intersection point is $(-1,-1,0)$.

Example 10: Find the point of intersection of the plane with equation $-27x-19y+7z=2$, and the line given by equations $x=1+2t$, $y=4-t$, and $z=3+5t$.

$$-27(1+2t)-19(4-t)+7(3+5t)=2$$
$$-27-54t-76+19t+21+35t=2$$
$$-35t+35t-82=2$$
$$-82=2 \text{ False!}$$

Line and plane do not intersect.

Note: normal vector to plane is

$\vec{n}=\langle -27,-19,7\rangle$

direction vector for line is

$\vec{v}=\langle 2,-1,5\rangle$

$\vec{n}\cdot\vec{v}=\langle -27,-19,7\rangle\cdot\langle 2,-1,5\rangle=-54+19+35$

$=-54+54=0$ ✓ as expected.

Angle between two planes:

The angle between two planes is the same as the angle between their normal vectors. Therefore, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, the angle between the two planes is described by this equation:

Note: angle θ between $\vec{u}$ an $\vec{v}$ is given by
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

Thus, the planes are

- orthogonal (perpendicular) when $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$
- parallel when $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

Example 11: Two planes have equations $3x + y - 4z = 3$ and $-9x - 3y + 12z = 4$. Determine if the planes are parallel, orthogonal, or neither. If neither, find the angle between them.

$\vec{n}_1 = \langle 3, 1, -4 \rangle$

$\vec{n}_2 = \langle -9, -3, 12 \rangle$

Notice: $\vec{n}_2 = -3\vec{n}_1$, so the planes are $\boxed{\text{parallel.}}$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\,\|\vec{n}_2\|} = \frac{|-27-3-48|}{\sqrt{9+1+16}\,\sqrt{81+9+144}} = \frac{|-78|}{\sqrt{26}\,\sqrt{234}} = \frac{78}{78} = 1$$

$\Rightarrow \theta = 0$

Example 12: Two planes have equations $x + y + z = 1$ and $x - 2y + 3z = 1$. Determine if the planes are parallel, orthogonal, or neither. If neither, find the angle between them. Find the equation of the line of intersection.

$\vec{n}_1 = \langle 1, 1, 1 \rangle$

$\vec{n}_2 = \langle 1, -2, 3 \rangle$ not parallel

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\,\|\vec{n}_2\|} = \frac{1-2+3}{\sqrt{1+1+1}\,\sqrt{1+4+9}} = \frac{2}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}} \Rightarrow \theta \approx 72.02^\circ$$

To find the line of intersection, we need a direction vector and a point:

Direction vector: $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 3+2, 1-3, -2-1 \rangle$

$= \langle 5, -2, -3 \rangle$

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Find a point that lines on both planes:

$x + y + z = 1$

$x - 2y + 3z = 1$

set $z = 0$: $\quad x + y = 1$

$\qquad\qquad\quad x - 2y = 1$

subtract eqns: $\quad 3y = 0$

$\qquad\qquad\qquad y = 0$

$y = 0, z = 0 \Rightarrow x + y + z = 1$ become $x = 1 \Rightarrow (1, 0, 0)$ pt on line

Direction vector is $\langle 5, -2, -3 \rangle$

Vector eqn for line:

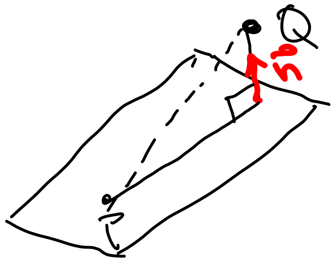
$$\langle x, y, z \rangle = \langle 1, 0, 0 \rangle + t \langle 5, -2, -3 \rangle$$

Parametric eqns for line:

$$\boxed{\begin{aligned} x &= 1 + 5t \\ y &= -2t \\ z &= -3t \end{aligned}}$$

Distance between a point and a plane:

> Theorem: Suppose P is a point in the plane, $\mathbf{n}$ is normal to the plane, and the point Q is not in the plane. Then the distance between a plane and the point Q is
>
> $$D = \left\|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\right\| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$$



$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

$$\left\|\text{proj}_{\vec{n}}\overrightarrow{PQ}\right\| = \left\|\left(\frac{\overrightarrow{PQ}\cdot\vec{n}}{\|\vec{n}\|^2}\right)\vec{n}\right\|$$

Example 13: Find the distance between the point $\underset{}{\overset{Q}{(1,3,-1)}}$ and the plane with equation $3x-4y+5z=6$.

$\vec{n} = \langle 3,-4,5\rangle$

Find any point in the plane.

Let $y=0,\ z=0$: Then $3x=6$, $x=2$

Point is $P=(2,0,0)$

$$D = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|\langle 1-2, 3-0, -1-0\rangle\cdot\langle 3,-4,5\rangle|}{\|\langle 3,-4,5\rangle\|} = \frac{|\langle -1,3,-1\rangle\cdot\langle 3,-4,5\rangle|}{\sqrt{9+16+25}} = \frac{|-3-12-5|}{\sqrt{50}} = \frac{20}{\sqrt{50}} = \frac{20}{\sqrt{25\cdot 2}} = \frac{20}{5\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = \boxed{2\sqrt{2}}$$

Example 14: Find the distance between the point (x_0, y_0, z_0) and the plane with equation $ax+by+cz+d=0$.

$\vec{n} = \langle a,b,c\rangle$ Suppose $P=(x,y,z)$ is a pt. in the plane.

$\overrightarrow{PQ} = \langle x_0-x,\ y_0-y,\ z_0-z\rangle$

$$D = \frac{|\overrightarrow{PQ}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{|a(x_0-x)+b(y_0-y)+c(z_0-z)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0 - ax + by_0 - by + cz_0 - cz|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_0+by_0+cz_0 \underbrace{-(ax+by+cz)}_{-d}|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

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So distance between plane $ax+by+cz+d=0$ and point $Q(x_0, y_0, z_0)$ is

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

Example 15: Verify that the planes given by $2x+3y-4z=2$ and $4x+6y-8z=27$ are parallel. Then find the distance between them.

See Summer 2015 notes in Archives

Distance between a point and a line in $\mathbb{R}^3$:

Theorem: The distance between a point Q and a line in $\mathbb{R}^3$ is

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

where $\mathbf{u}$ is a direction vector for the line and P is a point on the line.

Example 16: Find the distance between the point $Q(1,-2,4)$ and the line given by $x=2t$, $y=t-3$, $z=2t+2$.

Homework Qs

11.3 # 29 $\vec{u} = \hat{i} + 2\hat{j} + 2\hat{k} = \langle 1,2,2 \rangle$

Find direction cosines and direction angles

Recall: $\cos\alpha = \dfrac{\hat{i}\cdot\vec{u}}{\|\hat{i}\|\,\|\vec{u}\|}$

$$\cos\alpha = \frac{1}{\|\vec{u}\|} = \frac{1}{\sqrt{1+4+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

$\alpha = \cos^{-1}\left(\frac{1}{3}\right) \leftarrow$ angle between $\vec{u}$ and x-axis

$$\cos\beta = \frac{2}{\|\vec{u}\|} = \frac{2}{3}$$

$\beta = \cos^{-1}\left(\frac{2}{3}\right) \leftarrow$ angle between $\vec{u}$ and y-axis

$$\cos\gamma = \frac{2}{\|\vec{u}\|} = \frac{2}{3}$$

$\gamma = \cos^{-1}\left(\frac{2}{3}\right) \leftarrow$ angle between $\vec{u}$ and z-axis

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2$$
$$= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{9}{9} = 1 \checkmark$$

11.3 # 35

$\vec{u} = \langle 6,7 \rangle$, $\vec{v} = \langle 1,4 \rangle$

(a) Find $\text{proj}_{\vec{v}}\vec{u}$

(b) Find vector component of $\vec{u}$ that is orthogonal to $\vec{v}$

(a) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$

$$\vec{w}_1 = \text{proj}_{\vec{v}}\vec{u} = \left(\frac{\langle 6,7\rangle \cdot \langle 1,4\rangle}{\|\langle 1,4\rangle\|^2}\right)\langle 1,4\rangle = \left(\frac{6+28}{(\sqrt{1+16})^2}\right)\langle 1,4\rangle$$
$$= \left(\frac{34}{17}\right)\langle 1,4\rangle = 2\langle 1,4\rangle = \boxed{\langle 2,8\rangle}$$

(b) Vector component of $\vec{u}$ orth. to $\vec{v}$ is

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = \langle 6,7\rangle - \langle 2,8\rangle = \boxed{\langle 4,-1\rangle}$$