

11.6: Surfaces in Space

You should be familiar with 3 types of surfaces in $\mathbb{R}^3$:

1. Cylindrical surfaces
2. Quadric surfaces
3. Surfaces of revolution

Cylindrical surfaces (cylinders):

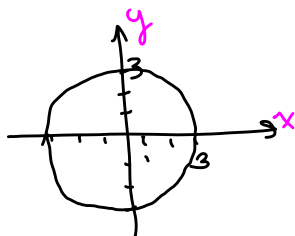
Definition:

Suppose C is a curve in a plane, and L is a line not in a parallel plane. The set of all lines parallel to L and intersecting C is a *cylindrical surface*, or *cylinder*.

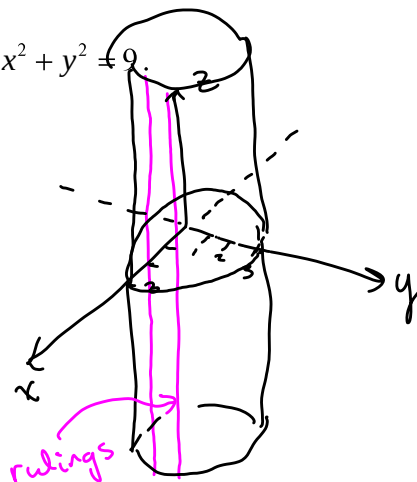
The curve C is the *generating curve* (or the *directrix*) of the cylinder, and the parallel lines are the *rulings*.

In a *right cylinder*, the rulings are perpendicular to the plane containing C . In this class, we'll stick to right cylinders in which C is contained in one of the coordinate planes. (Thus, the rulings are parallel to one of the coordinate axes.)

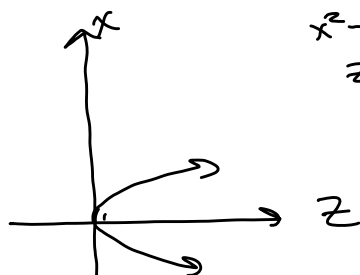
Example 1: Sketch the cylinder generated by the curve $x^2 + y^2 = 9$.



Circle w/radius 3,
center (0,0)

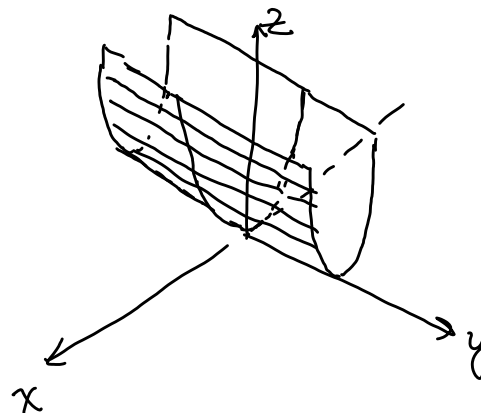


Example 2: Sketch the cylinder generated by the curve $x^2 - z = 0$.

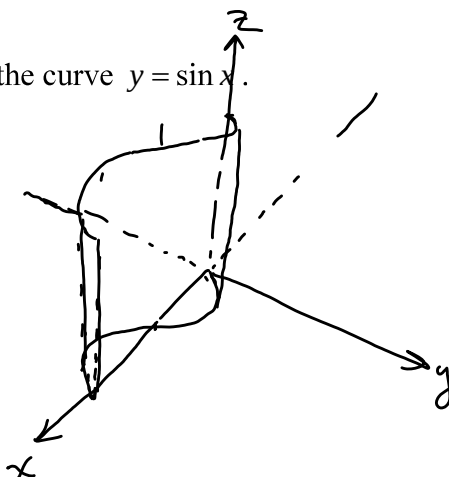
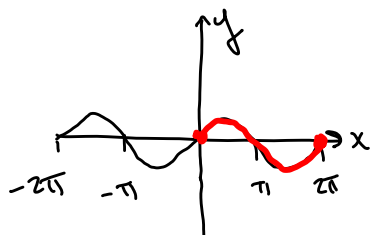


$$x^2 - z = 0$$

$$z = x^2$$



Example 3: Sketch the cylinder generated by the curve $y = \sin x$.



Example 4: Sketch the cylinder generated by the curve $x^2 + y^2 - 2x = 0$.

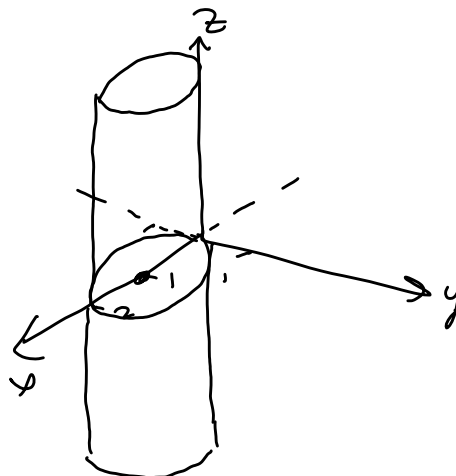
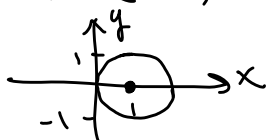
$$x^2 - 2x + y^2 = 0$$

complete square:

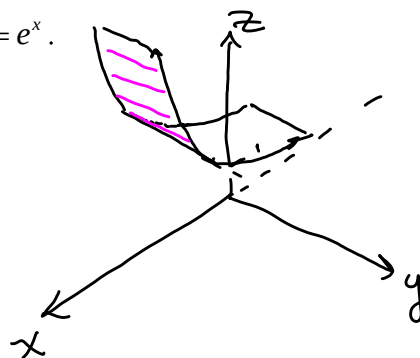
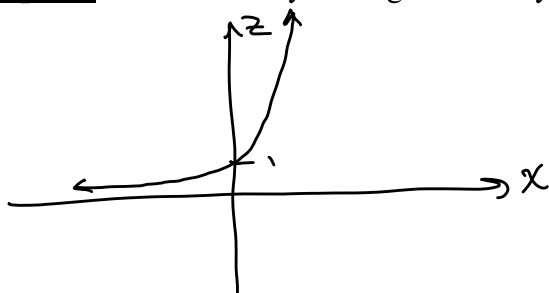
$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1$$

Circle with radius 1
and center (1, 0)



Example 5: Sketch the cylinder generated by the curve $z = e^x$.



Quadric surfaces:

Quadric surfaces in $\mathbb{R}^3$ are the three-dimensional analogs of conic sections in $\mathbb{R}^2$.

The *trace* of a surface in a plane is the intersection of the surface with the plane. The traces of the quadric surfaces are *conics* (ellipses, hyperbolas, and parabolas).

generally
A quadric surface in $\mathbb{R}^3$ is described by a second-degree equation in three variables. The general form of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

In this class, we'll stick to surfaces that are oriented in the same direction as one of the coordinate axes (no rotated conics!).

6 basic types of quadric surfaces:

1. Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

See p. 796

2. Hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

3. Hyperboloid of two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

4. Elliptic cone:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

5. Elliptic paraboloid:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$y = \frac{x^2}{a^2} + \frac{z^2}{c^2}$$

$$x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

6. Hyperbolic paraboloid:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$y = \frac{x^2}{a^2} - \frac{z^2}{c^2}$$

$$x = \frac{y^2}{b^2} - \frac{z^2}{c^2}$$

$$z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$y = -\frac{x^2}{a^2} + \frac{z^2}{c^2}$$

$$x = -\frac{y^2}{b^2} + \frac{z^2}{c^2}$$

See Pages 796–797 for accurate pictures of quadric surfaces. To practice sketching them by hand, see the worksheet handout. Trace it if you need to. Trust me, it helps! (Sincere appreciation to Mr. Bob Collings for the good hand drawings.)

Example 6: Sketch the surface described by the given equation.

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

Ellipsoid:

x-direction: $\sqrt{4} = 2$

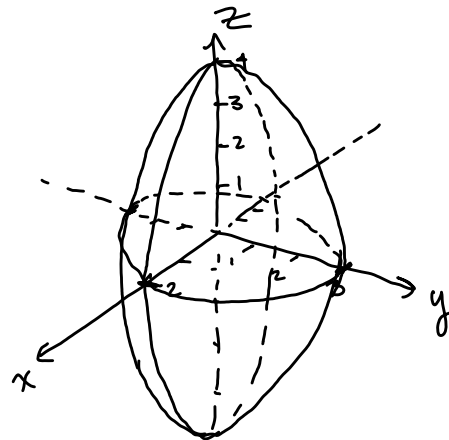
y-direction: $\sqrt{9} = 3$

z-direction: $\sqrt{16} = 4$

Traces:
Trace in yz-plane
 $x=0 \Rightarrow \frac{y^2}{9} + \frac{z^2}{16} = 1$
ellipse

Trace in xz-plane
 $y=0 \Rightarrow \frac{x^2}{4} + \frac{z^2}{16} = 1$
ellipse

Trace in xy-plane
 $z=0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$



Example 7: Sketch the surface described by the given equation.

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$$

$z=1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{1}{16} = 1$
 $\frac{1}{16} \left(\frac{x^2}{4} + \frac{y^2}{9} \right) = \frac{17}{16} \left(\frac{16}{17} \right)$
bigger ellipse

Trace in xy-plane:

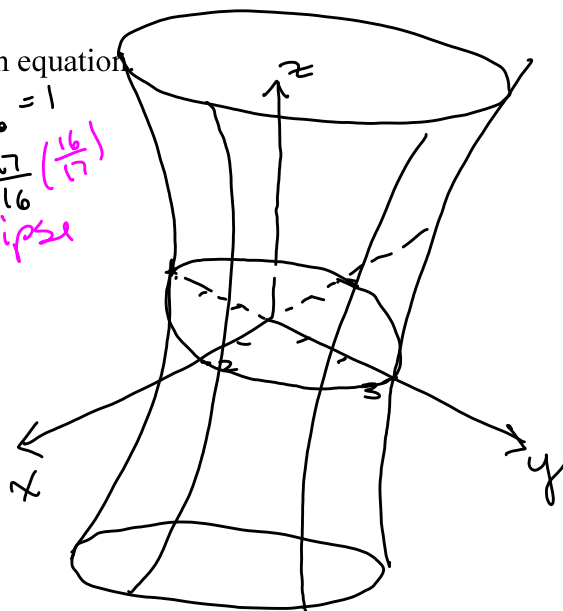
$z=0 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$ ellipse

Trace in yz-plane

$x=0 \Rightarrow \frac{y^2}{9} - \frac{z^2}{16} = 1$ hyperbola

Trace in xz-plane

$y=0 \Rightarrow \frac{x^2}{4} - \frac{z^2}{16} = 1$ hyperbola



Example 8: Sketch the surface described by the given equation.

$$\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{16} = 1$$

Trace in xy-plane

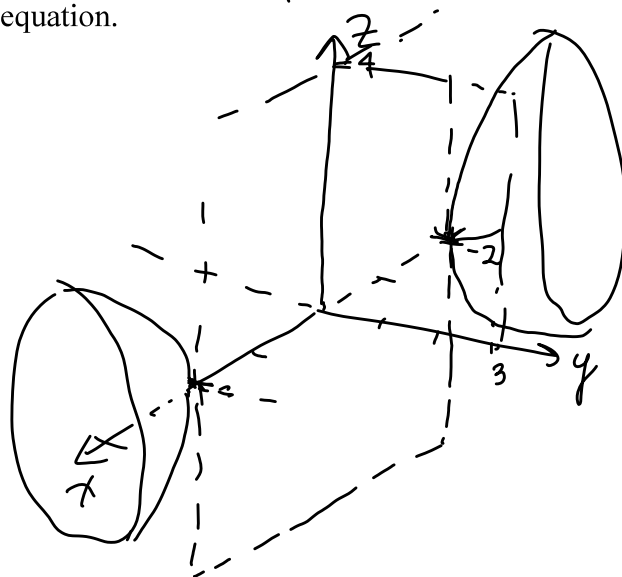
$z=0 \Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = 1$ hyperbola

Trace in yz-plane

$x=0 \Rightarrow -\frac{y^2}{9} - \frac{z^2}{16} = 1$ impossible
empty set

Trace in xz-plane

$y=0 \Rightarrow \frac{x^2}{4} - \frac{z^2}{16} = 1$ hyperbola



Example 9: Sketch the surface described by the given equation.

$$z^2 - x^2 - y^2 = 1$$

Example 10: Sketch the surface described by the given equation.

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 0$$

Example 11: Sketch the surface described by the given equation.

$$z = x^2 + y^2$$

Example 12: Sketch the surface described by the given equation.

$$z = x^2 - y^2$$

Example 13: Sketch the surface described by the given equation.

$$z = y^2 - x^2$$

Surfaces of revolution:

You found volumes of solids of revolution in Calculus 1 or Calculus 2; you found surface area of a surface of revolution. But what is the equation in $\mathbb{R}^3$ of the surface?

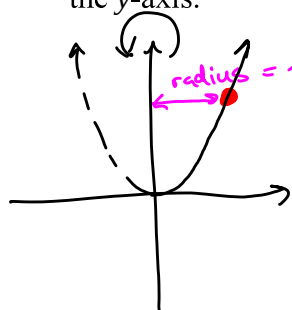
To find the equation of a surface of revolution, we need a radius function. The radius function will be a function of one variable only. The equation will take one of these forms:

Revolved about the x -axis: $y^2 + z^2 = [r(x)]^2$

Revolved about the y -axis: $x^2 + z^2 = [r(y)]^2$

Revolved about the z -axis: $x^2 + y^2 = [r(z)]^2$

Example 14: Find an equation for the surface generated by revolving the graph of $y = x^2$ about the y -axis.



write x in terms of y

$$y = x^2$$

$$x = \pm\sqrt{y} \quad (\text{choose } x = +\sqrt{y})$$

$$\text{radius} = \sqrt{y}$$

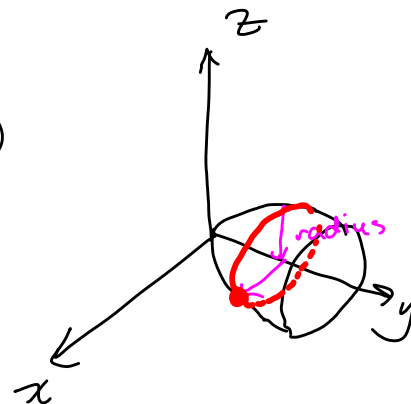
Egn will

take the form

$$x^2 + z^2 = [r(y)]^2$$

$$x^2 + z^2 = (\sqrt{y})^2$$

$$\boxed{x^2 + z^2 = y}$$



Example 15: Find a curve that will generate the surface $x^2 + z^2 - y^2 = 1$. Specify the axis of revolution.