12.1: Vector-Valued Functions

In $\mathbb{R}^2$, a *plane curve* can be described using parametric equations $x = f(t)$ and $y = g(t)$, where f and g are continuous functions of t on an interval I. (See Section 10.2 for a review.)

In $\mathbb{R}^3$, a *space curve* can be described using parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$, where f, g, and h are continuous functions of t on an interval I.

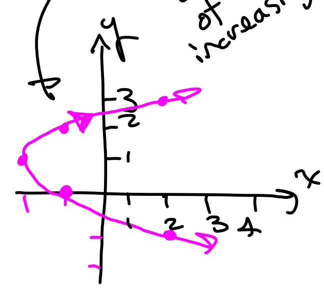
The plane curve or space curve is defined to be the *graph* (the set of ordered pairs or ordered triples) along with the defining parametric equations.

Note: The same graph can be generated by different sets of parametric equations. These are considered different curves, even though their graphs are the same.

Example 1: Sketch the graph defined by $x(t) = t^2 - 2$ and $y(t) = 1 + t$.

t	$x = t^2 - 2$	$y = 1+t$	
-2	$(-2)^2 - 2 = 2$	$1 - 2 = -1$	$(2,-1)$
-1	$(-1)^2 - 2 = -1$	$1 - 1 = 0$	$(-1,0)$
0	$0^2 - 2 = -2$	$1 + 0 = 1$	$(-2,1)$
1	-1	$1+1 = 2$	$(-1,2)$
2	2	$1+2 = 3$	$(2,3)$

must have arrow to indicate direction of increasing t



Example 2: Compare the graph defined by $x(t) = 2\sin t$ and $y(t) = 3\cos t$ with the graph defined by $x(t) = 2\cos 2t$ and $y(t) = 3\sin 2t$.

Real-valued functions: Output is a real number.
Vector-valued functions: Output is a vector.

<u>Definition</u>: A function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} = \langle f(t), g(t) \rangle \qquad \text{(in } \mathbb{R}^2\text{)}$$

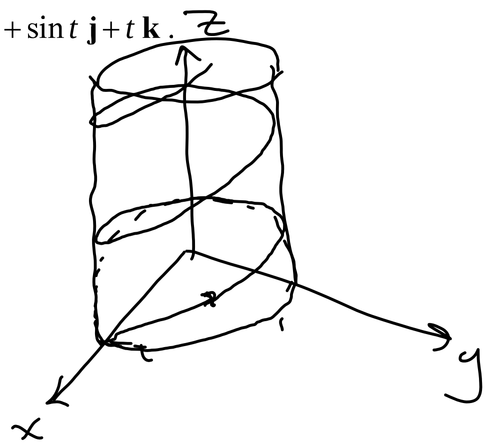
or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \langle f(t), g(t), h(t) \rangle \qquad \text{(in } \mathbb{R}^3\text{)}$$

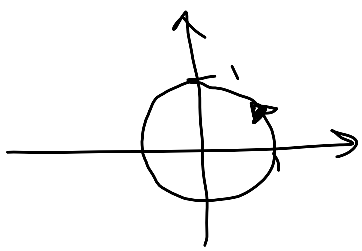
is called a *vector-valued function*. The component functions *f*, *g*, and *h* are real-valued functions of the parameter *t*.

When working with vector-valued functions, we usually think of each output as a *position vector*. The initial point of a position vector is the origin; the terminal point is a point on a curve. Thus, as *t* increases, the vector-valued function traces the graph of a curve.

Example 3: Sketch the curve represented by $\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$.



Projection onto xy-plane:



$\cos^2 t + \sin^2 t = 1$
$x^2 + y^2 = 1$

Example 4: What curve is represented by $\mathbf{r}(t) = \langle 3 - t, 4 + 2t, 6 - 3t \rangle$?

$$\left.\begin{aligned} x &= 3 - t \\ y &= 4 + 2t \end{aligned}\right\} \text{Parametric eqns for a line}$$
$$z = 6 - 3t$$

$\boxed{\text{This is a line.}}$

Example 5: Sketch the curve represented by $\mathbf{r}(t) = (2t-1)\,\mathbf{i} + (t^2+1)\,\mathbf{j} + (4-t^2)\,\mathbf{k}$.

$\vec{r}(-3) = \langle 2(-3)-1,\ (-3)^2+1,\ 4-(-3)^2 \rangle$
$= \langle -7, 10, -5 \rangle$

$\vec{r}(-2) = \langle -5, 5, 0 \rangle$

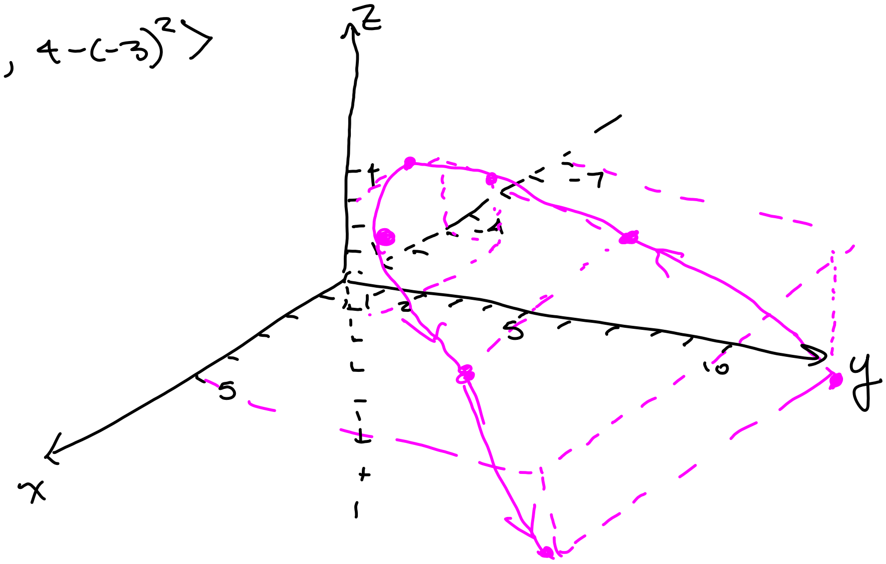
$\vec{r}(-1) = \langle -3, 2, 3 \rangle$

$\vec{r}(0) = \langle -1, 1, 4 \rangle$

$\vec{r}(1) = \langle 1, 2, 3 \rangle$

$\vec{r}(2) = \langle 3, 5, 0 \rangle$

$\vec{r}(3) = \langle 5, 10, -5 \rangle$



Domain of vector-valued functions:

For a value of t to be in the domain of a vector-valued function, it needs to be in the domain of <u>all</u> the component functions.

Example 6: Find the domain of $\mathbf{s}(t) = \left\langle \ln(t+1), \sin(t), \dfrac{1}{t-2} \right\rangle$.

$x(t) = \ln(t+1)$
$\Rightarrow t+1 > 0$
$t > -1$

$y(t) = \sin(t)$
all real #s

$z(t) = \dfrac{1}{t-2}$
$t \neq 2$

$-1 \qquad 2$

$\boxed{\text{Domain: } (-1,2) \cup (2,\infty)}$

Example 7: Find the domain of $\mathbf{s}(t) = \left\langle \sqrt{4-t}, e^t, \dfrac{3}{t} \right\rangle$.

$x(t) = \sqrt{4-t}$
$4-t \geq 0$
$4 \geq t$
$t \leq 4$

$y(t) = e^t$
all real #s

$z(t) = \dfrac{3}{t}$
$t \neq 0$

$0 \qquad 4$

$\boxed{\text{Domain: } (-\infty,0) \cup (0,4]}$

Representing a curve by a vector-valued function:

Example 8: Represent the plane curve $y = 4 - x^2$ by a vector-valued function.

Let $x = t$

Then $y = 4 - t^2$

$\vec{r}(t) = \langle t, 4 - t^2 \rangle$

or $\vec{r}(t) = t\,\hat{i} + (4 - t^2)\,\hat{j}$

Example 9: Represent the plane curve $x^2 + \dfrac{y^2}{5} = 1$ by a vector-valued function.

For an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, you can let $x = a\cos t$, $y = b\sin t$

Note: $\dfrac{(a\cos t)^2}{a^2} + \dfrac{(b\sin t)^2}{b^2} = 1$

$\cos^2 t + \sin^2 t = 1$ True

Our ellipse:

Let $x = a\cos t = 1\cos t$

$y = b\sin t = \sqrt{5}\sin t$

$\boxed{\vec{r}(t) = \langle \cos t, \sqrt{5}\sin t \rangle}$

Example 10: Find a vector-valued function that represents the line segment joining points $P(3, 4, -2)$ and $Q(-4, -3, -1)$.

$\overrightarrow{PQ} = \langle -4-3, -3-4, -1-(-2) \rangle = \langle -7, -7, 1 \rangle$

$\vec{r}(t) = \langle 3, 4, -2 \rangle + t\langle -7, -7, 1 \rangle$

$\boxed{\vec{r}(t) = \langle 3 - 7t, 4 - 7t, -2 + t \rangle}$

Note: $t = 0 \Rightarrow \vec{r}(0) = \langle 3 - 7(0), 4 - 7(0), -2 + 0 \rangle$

$= \langle 3, 4, -2 \rangle$ Position vector for Point $P(3, 4, -2)$

(Vector from origin to P)

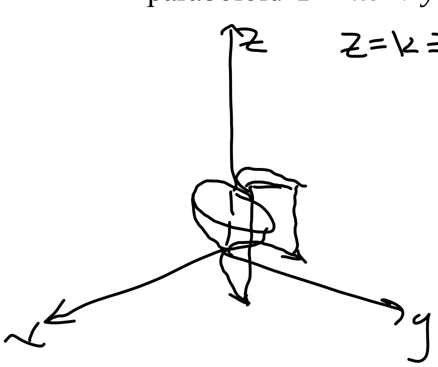
$t = 1 \Rightarrow \vec{r}(1) = \langle 3-7, 4-7, -2+1 \rangle = \langle -4, -3, -1 \rangle$

Position vector for Point $Q(-4, -3, -1)$

Example 11: Find a vector-valued function that represents the curve of intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$.

$z = k \Rightarrow k = 4x^2 + y^2$

ellipses



Let $x = t$.

Then $y = x^2 \Rightarrow y = t^2$

substitute $y = t^2$, $x = t$ into $z = 4x^2 + y^2$

$z = 4t^2 + (t^2)^2$

$= 4t^2 + t^4$

$\boxed{\vec{r}(t) = \langle t, t^2, 4t^2 + t^4 \rangle}$

Limits:

<u>Definition: Limit of a Vector-Valued Function</u>

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, then $\lim\limits_{x \to a} \mathbf{r}(t) = \left\langle \lim\limits_{x \to a} f(t), \lim\limits_{x \to a} g(t) \right\rangle$, provided these limits of the component functions exist.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\lim\limits_{x \to a} \mathbf{r}(t) = \left\langle \lim\limits_{x \to a} f(t), \lim\limits_{x \to a} g(t), \lim\limits_{x \to a} h(t) \right\rangle$, provided these limits of the component functions exist.

The properties of limits (sum, difference, scalar multiples, etc.) for vector-valued functions are similar to those for real-valued functions.

<u>Note</u>: The limit of a vector-valued function is a <u>vector</u>.

<u>Example 12:</u> Suppose that $\mathbf{r}(t) = \dfrac{\sin t}{t}\,\mathbf{i} + \dfrac{t^2 - 1}{2t^2 + 1}\,\mathbf{j} + \cos(t)\,\mathbf{k}$. Find $\cancel{\lim\limits_{x \to 0}}\,\mathbf{r}(t)$. $\lim\limits_{t \to 0} \vec{r}(t)$

$$\lim_{t \to 0} \vec{r}(t) = \left\langle 1, \frac{-1}{1}, \cos(0) \right\rangle = \boxed{\langle 1, -1, 1 \rangle}$$

Note:

$$\lim_{t \to 0} \left(\frac{\sin t}{t} \right) \overset{H}{=} \lim_{t \to 0} \left(\frac{\cos t}{1} \right) = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

<u>Example 13:</u> Determine $\lim\limits_{\cancel{x \to 0}\; t \to 0} \left\langle \dfrac{e^t - 1}{t}, \dfrac{\sqrt{1+t} - 1}{t}, \dfrac{3}{1+t} \right\rangle$.

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$$\lim_{t \to 0} \left\langle \frac{e^t - 1}{t}, \frac{\sqrt{1+t} - 1}{t}, \frac{3}{1+t} \right\rangle = \lim_{t \to 0} \left\langle \frac{e^t}{1}, \frac{\frac{1}{2}(1+t)^{-1/2}(1)}{1}, \frac{3}{1+0} \right\rangle$$

$$= \left\langle \frac{e^0}{1}, \frac{1}{2\sqrt{1+0}}, 3 \right\rangle = \boxed{\left\langle 1, \frac{1}{2}, 3 \right\rangle}$$

Continuity:

> Definition:
>
> A vector-valued function **r** is *continuous at a point* given by $t = a$ if $\lim_{t \to a} \mathbf{r}(t)$ exists and $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.
>
> A vector-valued function **r** is *continuous on an interval I* if it is continuous at every point in the interval.

Example 14: On what intervals is $\mathbf{r}(t) = \langle \tan(t), t, t^2 \rangle$ continuous?

$\vec{r}(t)$ is continuous except for $t = (2k+1)\frac{\pi}{2}$, k any integer.

Example 15: On what intervals is $\mathbf{r}(t) = \left\langle \sqrt{4-t^2}, \frac{1}{t}, \frac{t-3}{5} \right\rangle$ continuous?

$\sqrt{4-t^2}$ is defined for $-2 \le t \le 2$

$4 - t^2 \geq 0$

$4 \geq t^2$

$t^2 \leq 4$

Also $t \neq 0$ (because of $\frac{1}{t}$)

$\vec{r}$ is continuous on $(-2,0) \cup (0,2)$