12.2: Differentiation and Integration of Vector-Valued Functions

Differentiation:

Recall: CalcI def'n of derivative:
$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

Definition: The derivative of a vector-valued function **r** is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then $\vec{r}$ is differentiable at t. If **r** is differentiable for all t in an open interval I, then **r** is differentiable on the interval I.

For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$,

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{t}, \frac{g(t+\Delta t) - g(t)}{t}, \frac{h(t+\Delta t) - h(t)}{t} \right\rangle$$

$$= \left\langle \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{t}, \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{t}, \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{t} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

Theorem:

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, where f and g are differentiable functions of t, then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$.

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions of t, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.

Example 1: Suppose $\mathbf{r}(t) = 4\sqrt{t}\ \mathbf{i} + t^2\sqrt{t}\ \mathbf{j} + \ln(t^2)\ \mathbf{k}$. Find $\mathbf{r}'(t)$.

$\vec{r}(t) = \langle 4t^{1/2}, t^{5/2}, \ln(t^2) \rangle$

$\vec{r}'(t) = \langle 4(\frac{1}{2}t^{-1/2}), \frac{5}{2}t^{3/2}, \frac{1}{t^2}(2t) \rangle = \langle \frac{2}{\sqrt{t}}, \frac{5}{2}t^{3/2}, \frac{2}{t} \rangle$

Example 2: Suppose $\mathbf{r}(t) = \langle 7\cos t, 4\sin 2t \rangle$. Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, and $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$.

$\vec{r}'(t) = \langle -7\sin t, (4\cos 2t)(2) \rangle = \langle -7\sin t, 8\cos 2t \rangle$

$\vec{r}''(t) = \langle -7\cos t, (-8\sin(2t))(2) \rangle = \langle -7\cos t, -16\sin(2t) \rangle$

$\vec{r}'(t)\cdot\vec{r}''(t) = (-7\sin t)(-7\cos t) + (8\cos 2t)(-16\sin 2t)$

$= \boxed{49\sin t\cos t - 128\cos 2t\sin 2t}$

Definition:

The parametrization of the curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is considered *smooth* on an open interval I if

1. the component derivatives f', g', and h' are continuous on I

and

2. $\mathbf{r}'(t)$ is nonzero on all of I. $\vec{r}(t) \neq \vec{0}$

Example 3: Suppose $\mathbf{r}(t) = \sqrt{t}\ \mathbf{i} + (t^2 - 1)\ \mathbf{j} + \frac{1}{4}t\ \mathbf{k}$, $t \geq 0$. Find the open intervals on which the curve generated by the function is smooth.

$\vec{r}(t) = \langle t^{1/2}, t^2 - 1, \frac{1}{4}t \rangle, \ t \geq 0$

$\vec{r}'(t) = \langle \frac{1}{2}t^{-1/2}, 2t, \frac{1}{4} \rangle$

$= \langle \frac{1}{2\sqrt{t}}, 2t, \frac{1}{4} \rangle$

Note: $\frac{1}{2\sqrt{t}}$ is only defind for $t > 0$
(not defind for $t = 0$)

$\vec{r}$ is smooth for $t > 0$.

Example 4: Suppose $\mathbf{r}(t) = \langle 5\cos t - \cos 5t, 5\sin t - \sin 5t \rangle$, $0 \le t \le 2\pi$. Find the open intervals on which the curve generated by the function is smooth.

$\vec{r}\,'(t) = \langle -5\sin t + 5\sin 5t, 5\cos t - 5\cos 5t \rangle$

where is x-component 0?

$-5\sin t + 5\sin 5t = 0$

$t = 0,\ t = \pi,\ t = 2\pi$

$t = \frac{\pi}{2} \Rightarrow -5\sin\frac{\pi}{2} + 5\sin\frac{5\pi}{2}$

$-5(1) + 5(1) = 0$

$t = \frac{3\pi}{2} \Rightarrow -5\sin\frac{3\pi}{2} + 5\sin\frac{15\pi}{2}$

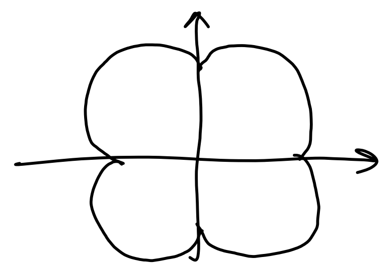
$-5(-1) + 5(-1)$

$= 0$

$5\cos t - 5\cos 5t = 0$

for $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

Also.



$\boxed{\text{Not smooth for } t = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}, 2\pi}$

Properties of the Derivative:

1. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\frac{d}{dt}[\mathbf{r}(t) + \mathbf{u}(t)] = \mathbf{r}'(t) + \mathbf{u}'(t)$

3. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

4. $\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)] = \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t)$

5. $\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)] = \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t)$

product rules

6. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$ (chain rule)

7. If $\mathbf{r}(t)\cdot\mathbf{r}(t) = c$, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$

Why is Prop #7 true?

Suppose $\vec{r}(t)\cdot\vec{r}(t) = c$

Let's calculate $\vec{r}(t)\cdot\vec{r}\,'(t)$

From Prop #4,

$\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)]$

$= \vec{r}(t)\cdot\vec{r}\,'(t) + \vec{r}\,'(t)\cdot\vec{r}(t)$

$= 2\vec{r}(t)\cdot\vec{r}\,'(t)$

Therefore,

$\frac{1}{2}\frac{d}{dt}[\vec{r}(t)\cdot\vec{r}(t)] = \vec{r}(t)\cdot\vec{r}\,'(t)$

$\frac{1}{2}\frac{d}{dt}(c) = \vec{r}(t)\cdot\vec{r}\,'(t)$

$\frac{1}{2}(0) = \vec{r}(t)\cdot\vec{r}\,'(t)$

So, $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$

Integration:

italics

Definition:

Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and that f, g, and h are continuous on $[a,b]$. Then,

$$\int \mathbf{r}(t)\, dt = \left\langle \int f(t)\, dt, \int g(t)\, dt, \int h(t)\, dt \right\rangle$$

and

$$\int_a^b \mathbf{r}(t)\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle .$$

Note: The constants of integration for the components can be combined into a vector **c**.

Example 5: Calculate $\int \left(t^2\, \mathbf{i} + \cos t\, \mathbf{j} + 5\, \mathbf{k} \right) dt$.

$$\int \langle t^2, \cos t, 5 \rangle\, dt = \left\langle \int t^2\, dt, \int \cos t\, dt, \int 5\, dt \right\rangle$$

$$= \left\langle \frac{t^3}{3} + c_1, \sin t + c_2, 5t + c_3 \right\rangle$$

$$= \left\langle \frac{t^3}{3}, \sin t, 5t \right\rangle + \langle c_1, c_2, c_3 \rangle$$

$$= \boxed{\frac{t^3}{3}\, \hat{i} + \sin t\, \hat{j} + 5t\, \hat{k} + \vec{c}}$$

Example 6: Calculate $\int_1^2 \left(\ln t\, \mathbf{i} + \frac{1}{t+2}\, \mathbf{j} + (t-1)^2\, \mathbf{k} \right) dt$.

see summer notes

Example 7: Suppose $\mathbf{r}'(t) = \langle e^{3t}, \sin(4t), 4t^3 \rangle$ and $\mathbf{r}(0) = \langle 3, -2, 1 \rangle$. Find $\mathbf{r}(t)$.

This is a vector-valued differential eqn with an initial condition

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \int \langle e^{3t}, \sin(4t), 4t^3 \rangle\,dt$$

$$= \left\langle \frac{1}{3}e^{3t}, -\frac{1}{4}\cos(4t), \frac{4t^4}{4} \right\rangle + \vec{C}$$

$$\vec{r}(0) = \langle 3, -2, 1 \rangle \Rightarrow \left\langle \frac{1}{3}e^{3(0)}, -\frac{1}{4}\cos(4(0)), 0^4 \right\rangle + \vec{C} = \langle 3, -2, 1 \rangle$$

$$\left\langle \frac{1}{3}, -\frac{1}{4}, 0 \right\rangle + \vec{C} = \langle 3, -2, 1 \rangle$$

$$\vec{C} = \langle 3, -2, 1 \rangle - \left\langle \frac{1}{3}, -\frac{1}{4}, 0 \right\rangle$$

$$= \left\langle \frac{9}{3} - \frac{1}{3}, -\frac{8}{4} + \frac{1}{4}, 1 - 0 \right\rangle = \left\langle \frac{8}{3}, -\frac{7}{4}, 1 \right\rangle$$

Final answer:

$$\vec{r}(t) = \left(\frac{1}{3}e^{3t} + \frac{8}{3}\right)\hat{i} + \left(-\frac{1}{4}\cos(4t) - \frac{7}{4}\right)\hat{j} + (t^4 + 1)\hat{k}$$

Example 8: Suppose $\mathbf{r}''(t) = -4\cos(t)\,\mathbf{j} - 3\sin(t)\,\mathbf{k}$, $\mathbf{r}'(0) = 2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{r}(0) = 4\,\mathbf{j}$. Find $\mathbf{r}(t)$.

$$\vec{r}\,''(t) = \langle 0, -4\cos(t), -3\sin(t) \rangle$$

$$\vec{r}\,'(t) = \int \vec{r}\,''(t)\,dt = \langle 0, -4\sin(t), 3\cos(t) \rangle + \vec{c}_1$$

$$\vec{r}\,'(0) = \langle 2, 0, 3 \rangle \Rightarrow \langle 0, -4\sin(0), 3\cos(0) \rangle + \vec{c}_1 = \langle 2, 0, 3 \rangle$$

$$\langle 0, 0, 3 \rangle + \vec{c}_1 = \langle 2, 0, 3 \rangle$$

$$\vec{c}_1 = \langle 2, 0, 3 \rangle - \langle 0, 0, 3 \rangle = \langle 2, 0, 0 \rangle$$

$$\vec{r}\,'(t) = \langle 2, -4\sin(t), 3\cos(t) \rangle$$

$$\vec{r}(t) = \int \vec{r}\,'(t)\,dt = \langle 2t, 4\cos(t), 3\sin(t) \rangle + \vec{c}_2$$

$$\vec{r}(0) = \langle 0, 4, 0 \rangle \Rightarrow \langle 2(0), 4\cos(0), 3\sin(0) \rangle + \vec{c}_2 = \langle 0, 4, 0 \rangle$$

$$\langle 0, 4, 0 \rangle + c_2 = \langle 0, 4, 0 \rangle$$

$$c_2 = \langle 0, 4, 0 \rangle - \langle 0, 4, 0 \rangle = \langle 0, 0, 0 \rangle = \vec{0}$$

$$\boxed{\vec{r}(t) = \langle 2t, 4\cos(t), 3\sin(t) \rangle}$$