12.3: Velocity and Acceleration

Suppose an object's position is given by the vector-valued function $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. Then the first and second derivatives represent the velocity and acceleration. Note that the velocity and acceleration are vectors that have both magnitude and direction.

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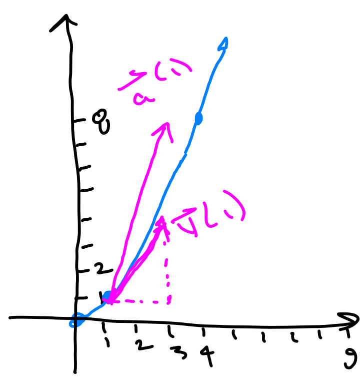
> Definition: Suppose x, y, and z are twice-differentiable functions of t, and $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Then the velocity vector, acceleration vector, and speed at time t are:
>
> Velocity = $\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$.
>
> Acceleration = $\vec{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle$.
>
> Speed = $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$.
>
> (This applies to vectors in $\mathbb{R}^2$ also.)

Note: The velocity vector is also known as the tangent vector. When its initial point is placed at the point on the curve, it is tangent to the curve.

Example 1: Suppose an object's position is given by the vector $\mathbf{r}(t) = \langle t^2, t^3\rangle$. Sketch the path of the object, and the velocity and acceleration vectors at the point $(1,1)$.

t	$x(t)=t^2$	$y(t)=t^3$	
0	0	0	(0,0)
1	1	1	(1,1)
2	4	8	(4,8)
3	9	27	(9,27)



$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, 3t^2\rangle$

$\vec{a}(t) = \vec{r}\,''(t) = \langle 2, 6t\rangle$

$\vec{v}(1) = \langle 2(1), 3(1)^2\rangle = \langle 2,3\rangle$

$\vec{a}(1) = \langle 2, 6(1)\rangle = \langle 2,6\rangle$

Example 2: Suppose an object's position is given by the vector $\mathbf{r}(t) = \langle t^2, t, t^{3/2}\rangle$. Find the velocity vector, the acceleration vector, and the speed in terms of t.

$\vec{v}(t) = \vec{r}\,'(t) = \boxed{\langle 2t, 1, \tfrac{3}{2}t^{1/2}\rangle}$

$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t) = \boxed{\langle 2, 0, \tfrac{3}{4}t^{-1/2}\rangle}$

Speed:

$\|\vec{v}(t)\| = \sqrt{(2t)^2 + (1)^2 + (\tfrac{3}{2}t^{1/2})^2} = \sqrt{4t^2 + 1 + \tfrac{9}{4}t}$

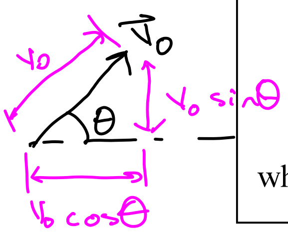
$= \sqrt{\dfrac{16t^2 + 4 + 9t}{4}} = \boxed{\dfrac{\sqrt{16t^2+4+9t}}{2}}$

Projectile motion:

italics

> Theorem:
> If a projectile is launched from an initial height h with an initial speed v_0 and angle of elevation θ, and if air resistance is neglected, then the projectile's path is described by the vector function
>
> $$\mathbf{r}(t) = \left\langle (v_0 \cos\theta)t, h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 \right\rangle,$$
>
> or italics
>
> where g is the acceleration of gravity.

$\|\vec{v}_0\| = v_0$



$$\vec{a}(t) = \langle 0, -g \rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 0, -gt \rangle + \vec{C}$$

$$\vec{v}(0) = \langle v_0 \cos\theta, v_0 \sin\theta \rangle$$

Thus, $\vec{v}(0) = \langle v_0 \cos\theta, v_0 \sin\theta \rangle = \langle 0, -g(0) \rangle + \vec{C}$

$$\langle v_0 \cos\theta, v_0 \sin\theta \rangle = \vec{C}$$

$$\vec{v}(t) = \langle 0, -gt \rangle + \vec{v}_0 = \langle 0, -gt \rangle + \langle v_0 \cos\theta, v_0 \sin\theta \rangle = \langle v_0 \cos\theta, v_0 \sin\theta - gt \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - \tfrac{1}{2}gt^2 \rangle + \vec{d}$$

Example 3: Suppose a projectile is fired at a height of 3000 feet above the ground with an initial velocity of 900 feet/second at an angle of 45 degrees above the horizontal. Determine the maximum height and range of the projectile.

$$\vec{r}(0) = \langle 0, h \rangle$$

$$\vec{r}(0) = \langle (v_0 \cos\theta)(0), (v_0 \sin\theta)0 - \tfrac{1}{2}g(0)^2 \rangle + \vec{d} = \langle 0, h \rangle$$

$$\therefore \vec{d} = \langle 0, h \rangle$$

So, $\vec{r}(t) = \langle t v_0 \cos\theta, t v_0 \sin\theta - \tfrac{1}{2}gt^2 + h \rangle$

$$\vec{r}(t) = \langle t v_0 \cos\theta, t v_0 \sin\theta - \tfrac{1}{2}gt^2 + h \rangle$$

$$= t\left(900 \tfrac{ft}{s} \cos 45^\circ\right)\hat{i} + t\left(900 \tfrac{ft}{s} \sin 45^\circ\right)\hat{j} - \tfrac{1}{2}gt^2\hat{j} + 3000\text{ft}\,\hat{j}$$

use $g = 32$ ft/sec²

$$x(t) = t v_0 \cos\theta = t\left(900 \tfrac{ft}{sec} \cdot \tfrac{\sqrt{2}}{2}\right) \approx 636.396 \text{ ft/sec } t$$

$$y(t) = t v_0 \sin\theta - \tfrac{1}{2}\left(\tfrac{32 ft}{s^2}\right)t^2 + 3000 \text{ft}$$

$$= t\left(900 \tfrac{ft}{sec} \cdot \tfrac{\sqrt{2}}{2}\right) - \tfrac{16 ft}{sec^2}t^2 + 3000 \text{ft} \approx 636.396 \tfrac{ft}{sec} t - \tfrac{16 ft}{sec^2} t^2 + 3000 \text{ft}$$

At max height, $y'(t) = 0 \Rightarrow y'(t) = 636.396 \tfrac{ft}{sec} - \tfrac{32 ft}{sec^2} t = 0$

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Ex 3 cont'd:

$$636.396 \frac{ft}{sec} = 32 \frac{ft}{sec^2} t$$

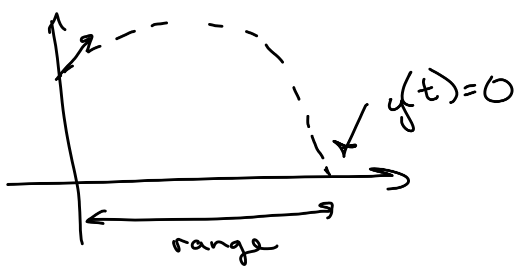
$$\frac{636.396 ft/sec}{32 ft/sec^2} = t$$

$$t \approx 19.887 \frac{ft}{sec} \cdot \frac{sec^2}{ft} \approx 19.887 \text{ sec}$$

Max height $= y(19.887 \text{ sec}) = 636.396 \frac{ft}{sec}(19.887 \text{ sec}) - 16 \frac{ft}{sec^2}(19.887 \text{ sec})^2 + 3000 ft$

$$\approx 9328.125 ft \approx \boxed{9328 ft}$$

To find range, we need to figure out what value of t results in $y(t) = 0$.



$$0 = y(t) = 636.396 \frac{ft}{sec} t - 16 \frac{ft}{sec^2} t^2 + 3000 ft$$

Use quadratic formula $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = -16$, $b = 636.396$, $c = 3000$

$$t = \frac{-636.396 ft/sec \pm \sqrt{(636.396 ft/s)^2 - 4(-16 ft/sec^2)(3000 ft)}}{2(-16 ft/sec^2)}$$

$$\approx 44.03 \text{ sec}$$

Put this into $x(t)$ to find range.

$$x(44.03 \text{ sec}) = 636.396 ft/sec \, (44.03 \text{ sec}) = \boxed{28\,021 ft}$$

Example 4: Suppose a projectile is fired from ground level at an angle of 12° above the horizontal. Determine the minimum initial velocity necessary for the projectile to have a range of 200 feet.

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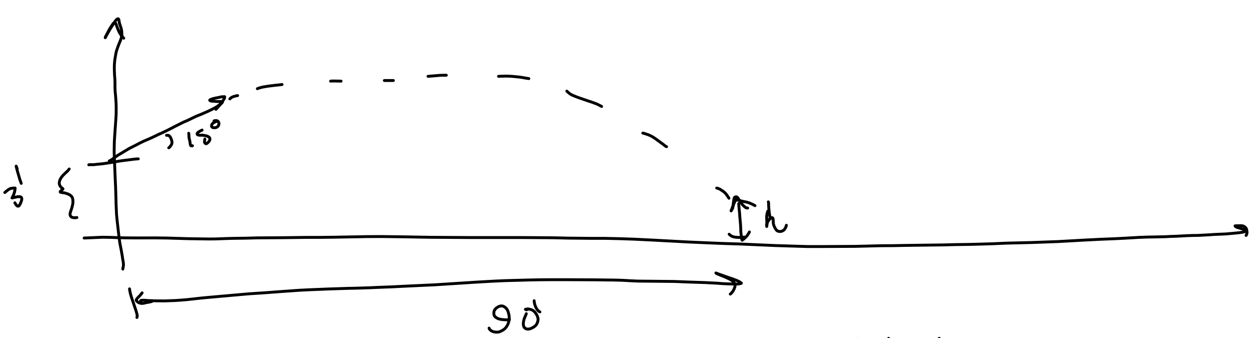
Example 5: A baseball player at second base throws a ball 90 feet to the player at first base. The ball is released at 3 feet above the ground with an initial velocity of 70 miles per hour, at an angle of 15° above the horizontal. At what height does the first baseman catch the ball?



$$v_0 = \frac{70\ \cancel{\text{miles}}}{\cancel{\text{hr}}}\left(\frac{5280\text{ ft}}{1\ \cancel{\text{mile}}}\right)\left(\frac{1\ \cancel{\text{hr}}}{60\ \cancel{\text{min}}}\right)\left(\frac{1\ \cancel{\text{min}}}{60\text{ sec}}\right) \approx 102.667\text{ ft/sec}$$

$$x(t) = (v_0\cos\theta)t = 102.667\cos 15^\circ\ t\ \text{ft/sec} \approx 99.17\ \text{ft/sec}\ t$$

$$y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2 + h_0 = 102.667\ \text{ft/sec}\ \sin 15^\circ t - 16\frac{\text{ft}}{\text{sec}^2}t^2 + 3\text{ ft}$$

$$= 26.57\frac{\text{ft}}{\text{sec}}t - \frac{16\text{ ft}}{\text{sec}^2}t^2 + 3\text{ ft}$$

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At first base, $x(t) = 90'$

$$(99.17 \text{ ft/sec})t = 90 \text{ ft}$$

$$t = \frac{90 \text{ ft}}{99.17 \text{ ft/sec}}$$

$$\approx 0.908 \text{ sec}$$

Put $t = 0.908$ sec into $y(t)$:

$$y(t) = 26.57 \frac{\text{ft}}{\text{sec}} t - \frac{16 \text{ ft}}{\text{sec}^2} t^2 + 3 \text{ ft}$$

$$y(0.908 \text{ sec}) = 26.57 \frac{\text{ft}}{\text{sec}} (0.908 \text{ sec}) - \frac{16 \text{ ft}}{\text{sec}^2} (0.908 \text{ sec})^2 + 3 \text{ ft}$$

≈ 13.9 ft (maybe didn't catch)