12.4: Tangent Vectors and Normal Vectors

The unit tangent vector:

> Definition: The Unit Tangent Vector
>
> Let C be a smooth curve represented by **r** on an open interval I. The unit tangent vector is
>
> $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \text{ provided } \mathbf{r}'(t) \neq \mathbf{0}.$$

Note: If r represents the position of a particle, then $\mathbf{r}'(t)$ is the velocity vector.

Example 1: Find the unit tangent vector for $\mathbf{r}(t) = (2\sin t)\,\mathbf{i} + (2\cos t)\,\mathbf{j} + (\cancel{4\sin^2 5})\,\mathbf{k}$ $4\sin^2 t\,\hat{k}$ at the point $P(1, \sqrt{3}, 1)$. Find a set of parametric equations for the line tangent to the space curve at point P.

$\vec{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle$

$\vec{r}\,'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(2\cos t)^2 + (-2\sin t)^2 + (8\sin t\cos t)^2} = \sqrt{4\cos^2 t + 4\sin^2 t + 64\sin^2 t\cos^2 t}$

$= \sqrt{4(\underbrace{\cos^2 t + \sin^2 t}_{1}) + 64\sin^2 t\cos^2 t} = \sqrt{4 + 64\sin^2 t\cos^2 t}$

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The principal unit normal vector:

There are infinitely many vectors orthogonal to the unit tangent vector $\mathbf{T}(t)$. One of them is $\mathbf{T}'(t)$.

Why is $\mathbf{T}(t) \perp \mathbf{T}'(t)$?

$\vec{T}(t) \cdot \vec{T}(t) = \|\vec{T}(t)\|^2 = (1)^2 = 1$ (it's a unit vector)

Property #7 of derivative $\Rightarrow$ if $\vec{r}(t)\cdot\vec{r}(t) = c$, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$

Because $\vec{T}(t)\cdot\vec{T}(t)$ is constant, then $\vec{T}(t)\cdot\vec{T}\,'(t) = 0$

and so $\vec{T}(t) \perp \vec{T}\,'(t)$

Ex. 1 cont'd:

we need to figure out what t produces the point $P(1, \sqrt{3}, 1)$.

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle}{\sqrt{4 + 64\sin^2 t\cos^2 t}}$$

$$\vec{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle$$

Set $\vec{r}(t) = \langle 1, \sqrt{3}, 1\rangle$.

$$x(t) = 2\sin t = 1$$
$$\sin t = \tfrac{1}{2} \Rightarrow t = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \text{etc}$$

$$y(t) = 2\cos t = \sqrt{3}$$
$$\cos t = \tfrac{\sqrt{3}}{2} \Rightarrow t = \tfrac{\pi}{6}, \tfrac{11\pi}{6} \text{ et}$$

$$z(t) = 4\sin^2 t = 1$$
$$\sin^2 t = \tfrac{1}{4}$$
$$\sin t = \pm\tfrac{1}{2}$$

$$\Bigg\} \quad t = \frac{\pi}{6}$$

Put $\frac{\pi}{6}$ into $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$:

$$\vec{r}\,'\left(\tfrac{\pi}{6}\right) = \left\langle 2\cos\tfrac{\pi}{6}, -2\sin\tfrac{\pi}{6}, 8\sin\tfrac{\pi}{6}\cos\tfrac{\pi}{6}\right\rangle$$
$$= \left\langle 2\left(\tfrac{\sqrt{3}}{2}\right), -2\left(\tfrac{1}{2}\right), 8\left(\tfrac{1}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)\right\rangle = \langle\sqrt{3}, -1, 2\sqrt{3}\rangle$$

$$\left\|\vec{r}\,'\left(\tfrac{\pi}{6}\right)\right\| = \sqrt{(\sqrt{3})^2 + (-1)^2 + (2\sqrt{3})^2} = \sqrt{3+1+4(3)} = \sqrt{16} = 4$$

$$\vec{T}\left(\tfrac{\pi}{6}\right) = \frac{\langle\sqrt{3}, -1, 2\sqrt{3}\rangle}{4} = \boxed{\left\langle \frac{\sqrt{3}}{4}, -\frac{1}{4}, \frac{\sqrt{3}}{2}\right\rangle}$$

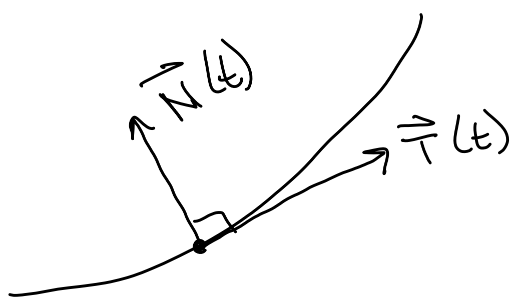
Find parametric eqns of line:

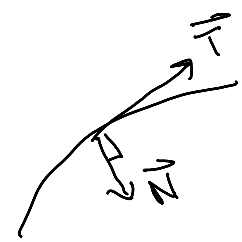
$$\langle x, y, z\rangle = \langle 1, \sqrt{3}, 1\rangle + t\left\langle \frac{\sqrt{3}}{4}, -\frac{1}{4}, \frac{\sqrt{3}}{2}\right\rangle$$

$$\boxed{x = 1 + \frac{t\sqrt{3}}{4}, \quad y = \sqrt{3} - \frac{t}{4}, \quad z = 1 + \frac{t\sqrt{3}}{2}}$$

If we normalize $\mathbf{T}'(t)$, we get the *principal unit normal vector*.

> Definition: Principal Unit Normal Vector
>
> Let C be a smooth curve represented by r on an open interval. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the principal unit normal vector at t is defined to be
>
> $$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$





Example 2: Calculate the unit tangent vector and the principal unit normal vector for $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 3t \rangle$. (in terms of t)

$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 3 \rangle$

$\|\vec{r}'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 3^2} = \sqrt{4\sin^2 t + 4\cos^2 t + 9} = \sqrt{13}$

(using $\sin^2 t + \cos^2 t = 1$)

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{\langle -2\sin t, 2\cos t, 3 \rangle}{\sqrt{13}} = \left\langle -\frac{2}{\sqrt{13}}\sin t, \frac{2}{\sqrt{13}}\cos t, 3 \right\rangle$

unit tangent vector.

$\vec{T}'(t) = \left\langle -\frac{2}{\sqrt{13}}\cos t, -\frac{2}{\sqrt{13}}\sin t, 0 \right\rangle$

$\|\vec{T}'(t)\| = \sqrt{\left(-\frac{2}{\sqrt{13}}\cos t\right)^2 + \left(-\frac{2}{\sqrt{13}}\sin t\right)^2 + 0^2} = \sqrt{\frac{4}{13}\cos^2 t + \frac{4}{13}\sin^2 t}$

$= \sqrt{\frac{4}{13}(\cos^2 t + \sin^2 t)} = \sqrt{\frac{4}{13}(1)} = \frac{\sqrt{4}}{\sqrt{13}} = \frac{2}{\sqrt{13}}$

$\vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \dfrac{\left\langle -\frac{2}{\sqrt{13}}\cos t, -\frac{2}{\sqrt{13}}\sin t, 0 \right\rangle}{\frac{2}{\sqrt{13}}} = \boxed{\langle -\cos t, -\sin t, 0 \rangle}$

Example 3: Calculate the unit tangent vector and the principal unit normal vector for $\mathbf{r}(t) = \langle t, t^2 \rangle$.

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}} = \boxed{\left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle = \vec{T}(t) \text{ [unit tangent vector]}}$$

Need to find $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

Use Product Rule for Scalar multiplication of a Vector (Derivative Property #3)

$$\vec{T}(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle = (1+4t^2)^{-1/2} \langle 1, 2t \rangle$$

$$\vec{T}\,'(t) = (1+4t^2)^{-1/2} \frac{d}{dt} \langle 1, 2t \rangle + \frac{d}{dt} (1+4t^2)^{-1/2} \langle 1, 2t \rangle$$

$$= (1+4t^2)^{-1/2} \langle 0, 2 \rangle - \frac{1}{2} (1+4t^2)^{-3/2} (8t) \langle 1, 2t \rangle$$

$$= \left\langle 0, \frac{2}{\sqrt{1+4t^2}} \right\rangle - \frac{4t}{(1+4t^2)^{3/2}} \langle 1, 2t \rangle$$

$$= \left\langle 0, \frac{2}{\sqrt{1+4t^2}} \right\rangle + \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, -\frac{8t^2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{\sqrt{1+4t^2}} \left(\frac{1+4t^2}{1+4t^2} \right) - \frac{8t^2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2+8t^2-8t^2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle$$

$$\|\vec{T}\,'(t)\| = \sqrt{\frac{16t^2}{(1+4t^2)^3} + \frac{4}{(1+4t^2)^3}} = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}} = \sqrt{\frac{4(4t^2+1)}{(1+4t^2)^3}}$$

$$= \sqrt{\frac{4}{(1+4t^2)^2}} = \frac{2}{1+4t^2}$$

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Ex 3 cont'd

Principal Unit Normal Vector $\quad \vec{N}(t) = \dfrac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$

$$= \frac{\left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle}{\frac{2}{1+4t^2}}$$

$$= \frac{1+4t^2}{2} \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}} \right\rangle = \vec{N}(t)$$

Note: $\vec{T}(t) = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle$

For plane curves, if $\vec{T}(t) = \langle x(t), y(t) \rangle$, then $\vec{N}(t)$ must be either

$$\vec{N}_1(t) = \langle -y(t), x(t) \rangle \quad \text{or} \quad \vec{N}_2(t) = \langle y(t), -x(t) \rangle$$

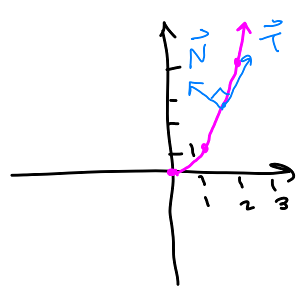
(Because $\vec{N}(t) \cdot \vec{T}(t) = 0$ and their magnitudes are both 1)

One of $\vec{N}_1$ or $\vec{N}_2$ will point toward the "inside" (concave side) of the curve. The other will point outward.

For $\vec{N}$, we want the one pointing inward.

$\vec{r}(t) = \langle t, t^2 \rangle$

t	x	y
0	0	0
1	1	1
2	2	4
3	3	9



Tangential and normal components of acceleration:

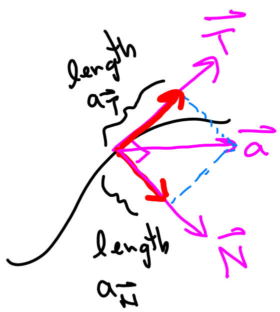
Recall: If $\mathbf{r}(t)\cdot\mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = c$, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = \mathbf{0}$.

Thus, if the velocity is constant, then the velocity and acceleration vectors must be orthogonal. In other words, if the speed $\|\mathbf{r}'(t)\|$ is constant, then $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = \mathbf{0}$.

If an object is not traveling at a constant speed, the velocity and acceleration vectors are not necessarily orthogonal.

The acceleration vector can be broken down into ~~the~~ two components: a tangential component acting in the direction of the line of motion, and a normal component acting perpendicular to the line of motion.

> Theorem: If $\mathbf{r}(t)$ is the position vector for a smooth curve, and if the principal unit normal vector $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the same plane as $\mathbf{T}(t)$ and $\mathbf{N}(t)$.



$$\vec a(t) = a_{\vec T}\vec T(t) + a_{\vec N}\vec N(t)$$

$$\text{scalars}\begin{cases} a_{\vec T} = \text{tangential component of acceleration} \\ a_{\vec N} = \text{normal component of acceleration} \end{cases}$$

This theorem follows from the fact that $\mathbf{a}(t)$ can be written as a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}(t)$. In other words,

$$\mathbf{a}(t) = a_{\mathrm{T}}\mathbf{T}(t) + a_{\mathrm{N}}\mathbf{N}(t).$$

a_{T} is the tangential component of acceleration; a_{N} is the normal component of acceleration.

> Theorem: If $\mathbf{r}(t)$ is the position vector for a smooth curve, and if $\mathbf{N}(t)$ exists, then the tangential and normal components of acceleration are as follows:
>
> $$a_{\mathrm{T}} = \frac{d}{dt}\|\mathbf{v}\| = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} = \frac{\vec a \cdot \vec r'}{\|\vec r'\|}$$
>
> $$a_{\mathrm{N}} = \|\mathbf{v}\|\|\mathbf{T}'\| = \mathbf{a}\cdot\mathbf{N} = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathrm{T}}^2}$$

The easiest method for finding $\mathbf{N}(t)$ is usually to calculate the other four values in $\mathbf{a}(t) = a_{\mathrm{T}}\mathbf{T}(t) + a_{\mathrm{N}}\mathbf{N}(t)$ and solve for $\mathbf{N}(t)$ algebraically.

Example 4: Suppose that $\mathbf{r}(t) = \langle t^3, 2t, 4t^2 \rangle$. Calculate $\mathbf{a}(t)$, a_{T}, a_{N}, $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for $t = 1$.

$$\vec{r}\,'(t) = \langle 3t^2, 2, 8t \rangle = \vec{v}(t)$$

$$\vec{r}\,'(1) = \langle 3(1)^2, 2, 8(1) \rangle = \langle 3, 2, 8 \rangle$$

$$\|\vec{r}\,'(1)\| = \sqrt{9+4+64} = \sqrt{77}$$

$$\vec{T}(1) = \frac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|} = \frac{\langle 3,2,8 \rangle}{\sqrt{77}} = \boxed{\left\langle \frac{3}{\sqrt{77}}, \frac{2}{\sqrt{77}}, \frac{8}{\sqrt{77}} \right\rangle = \vec{T}(1)}$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle 6t, 0, 8 \rangle$$

$$\vec{a}(1) = \vec{r}\,''(1) = \boxed{\langle 6, 0, 8 \rangle = \vec{a}(1)}$$

$$a_{\vec{T}}(1) = \vec{a}(1) \cdot \vec{T}(1) = \langle 6, 0, 8 \rangle \cdot \left\langle \frac{3}{\sqrt{77}}, \frac{2}{\sqrt{77}}, \frac{8}{\sqrt{77}} \right\rangle$$

$$= \frac{18}{\sqrt{77}} + 0 + \frac{64}{\sqrt{77}} = \boxed{\frac{82}{\sqrt{77}} = a_{\vec{T}}(1)}$$

$$a_{\vec{N}} = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|}$$

$$\vec{r}\,'(1) \times \vec{r}\,''(1) = \vec{v}(1) \times \vec{a}(1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 8 \\ 6 & 0 & 8 \end{vmatrix}$$

$$= \langle 16 - 0, 48 - 24, 0 - 12 \rangle = \langle 16, 24, -12 \rangle$$

$$\vec{a}_N = \frac{\|\langle 16, 24, -12 \rangle\|}{\|\langle 3, 2, 8 \rangle\|} = \frac{\sqrt{16^2 + 24^2 + 12^2}}{\sqrt{9+4+64}} = \boxed{\frac{\sqrt{976}}{\sqrt{77}} = a_{\vec{N}}}$$

$$\vec{a}(1) = a_{\vec{T}}(1)\vec{T}(1) + a_{\vec{N}}(1)\vec{N}(1)$$

$$\langle 6, 0, 8 \rangle = \frac{82}{\sqrt{77}} \left\langle \frac{3}{\sqrt{77}}, \frac{2}{\sqrt{77}}, \frac{8}{\sqrt{77}} \right\rangle + \frac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$$

$$= \left\langle \frac{246}{77}, \frac{164}{77}, \frac{656}{77} \right\rangle + \frac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$$

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$$\langle 6, 0, 8\rangle - \left\langle \frac{246}{77}, \frac{164}{77}, \frac{656}{77}\right\rangle = \frac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$$

$$\left\langle \frac{216}{77}, -\frac{164}{77}, -\frac{40}{77}\right\rangle = \frac{\sqrt{976}}{\sqrt{77}} \vec{N}(1)$$

$$\frac{\sqrt{77}}{\sqrt{976}} \left\langle \frac{216}{77}, -\frac{164}{77}, -\frac{40}{77}\right\rangle = \vec{N}(1)$$

$$\boxed{\left\langle \frac{216}{\sqrt{976}\sqrt{77}}, \frac{-164}{\sqrt{976}\sqrt{77}}, \frac{-40}{\sqrt{976}\sqrt{77}}\right\rangle = \vec{N}(1)}$$

$$\vec{N} = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$