13.10: Lagrange Multipliers

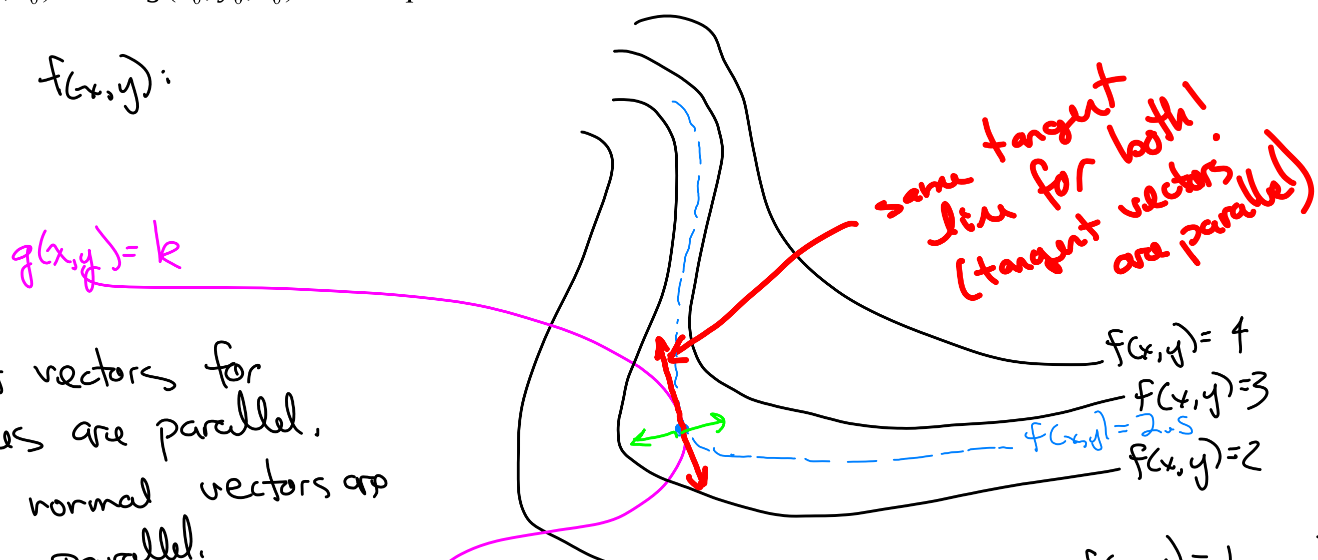
Suppose we want to maximize (or minimize) the function $f(x,y)$ while making sure that $g(x,y)=k$ is true. In this situation, the condition $g(x,y)=k$ is called a *constraint*.

Think of a sequence of level curves $f(x,y)=c_1$, $f(x,y)=c_2$, $f(x,y)=c_3$, etc. At the optimal value of c_n, the curve $f(x,y)=c_n$ will "barely touch" the curve $g(x,y)=k$. We want to find the points where $f(x,y)=c_n$ and $g(x,y)=k$ share a common tangent line. For the curves to share a common tangent line, their normal lines at the point of tangency $P(x_0,y_0)$ must be identical. Therefore, their gradients must be parallel.

We can write $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$, where λ is a constant called a Lagrange multiplier.

If we extend the concept to functions on $\mathbb{R}^3$, we optimize $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$. In this case, the optimal point is point of common tangency between some level surface $f(x,y,z)=c_n$ of $f(x,y,z)$, and the surface $g(x,y,z)=k$, which can be thought of as one of the level surfaces of $g(x,y,z)$. At the point of tangency, the gradient vectors $\nabla f(x_0,y_0,z_0)$ and $\nabla g(x_0,y_0,z_0)$ will be parallel.

maximize $f(x,y)$:



The tangent vectors for both curves are parallel. Thus, the normal vectors are also parallel.

At the min/max, ∇g is a scalar multiple of ∇f.

Lagrange's Theorem: Suppose the functions *f* and *g* have continuous first partial derivatives, and that *f* has an extremum at a point (x_0,y_0) on the smooth constraint curve $g(x,y)=k$.

lambda

If $\nabla g(x_0,y_0)\neq\mathbf{0}$, then there exists a real number λ such that $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$.

Note: This can be extended to functions of three variables, in which $f(x,y,z)$ has an extremum on the smooth level surface $g(x,y,z)=k$. In this case, there exists a real number λ such that $\nabla f(x_0,y_0,z_0)=\lambda\nabla g(x_0,y_0,z_0)$.

The Method of Lagrange Multipliers:

Suppose f and g satisfy the hypotheses of Lagrange's Theorem, that f has a maximum or minimum value satisfying $g(x,y,z)=k$, and that $\nabla g(x_0,y_0,z_0)\neq \mathbf{0}$.

Step 1: Find all values of λ, x, and y, and z such that $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ and $g(x,y,z)=k$.

This means you'll have to solve a system of simultaneous equations:

$$f_x(x,y,z)=\lambda g_x(x,y,z)$$
$$f_y(x,y,z)=\lambda g_y(x,y,z)$$
$$f_z(x,y,z)=\lambda g_z(x,y,z)$$
$$\cancel{f_z(x,y,z)=\lambda g_z(x,y,z)}$$
$$g(x,y,z)=k$$

Step 2: Evaluate f at each point obtained in Step 1. The largest of these values is the maximum, and the smallest of these values is the minimum.

Example 1: Find the extreme values of $f(x,y)=4x+6y$ on the circle $x^2+y^2=13$.

$\nabla f(x,y)=\langle 4,6\rangle$

$\nabla f=\lambda\nabla g$

$\langle 4,6\rangle=\lambda\langle 2x,2y\rangle$

$\Downarrow$ system of equations:

system of eqns
$$\begin{cases}4=\lambda(2x)\\ 6=\lambda(2y)\\ x^2+y^2=13\end{cases}\quad \Bigg\}\ \text{3 eqns in 3 variables}$$

$4=2\lambda x \Rightarrow \frac{4}{2x}=\lambda \Rightarrow \lambda=\frac{2}{x}$

$6=2\lambda y \Rightarrow \frac{6}{2y}=\lambda \Rightarrow \lambda=\frac{3}{y}$

Set λ's equal: $\frac{2}{x}=\frac{3}{y}$

$2y=3x$

$y=\frac{3}{2}x$

$g(x,y)=x^2+y^2$

$\nabla g(x,y)=\langle 2x,2y\rangle$

(If you wanted, you could define $g(x,y)=x^2+y^2-13$)

$x^2+y^2=13$

$x^2+\left(\frac{3}{2}x\right)^2=13$

$x^2+\frac{9}{4}x^2=13$

$\frac{4}{4}x^2+\frac{9}{4}x^2=13$

$\frac{13}{4}x^2=13$

$x^2=13\left(\frac{4}{13}\right)=4$

$x=\pm 2$

$x=2\Rightarrow y=\frac{3}{2}(2)=3$

$x=-2\Rightarrow y=\frac{3}{2}(-2)=-3$

So I have the points: $(2,3)$ and $(-2,-3)$

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Ex 1 cont'd:

$f(x,y) = 4x + 6y \Rightarrow$

$f(2,3) = 4(2) + 6(3) = 26$

$f(-2,-3) = 4(-2) + 6(-3) = -26$

So, the minimum is $f(-2,-3) = -26$, and the maximum is $f(2,3) = 26$.

Example 2: Find the maximum value of $P = xy^2z$ subject to the constraint $x + y + z = 32$.

$g(x,y,z) = x + y + z$

$\nabla g(x,y,z) = \langle 1, 1, 1 \rangle$

$P(x,y,z) = xy^2z$

$\nabla P(x,y,z) = \langle y^2z,\ 2xyz,\ xy^2 \rangle$

$\nabla P = \lambda \nabla g$

$\langle y^2z,\ 2xyz,\ xy^2 \rangle = \lambda \langle 1,1,1 \rangle$

$\Downarrow$

system
$$\begin{cases} y^2z = \lambda \\ 2xyz = \lambda \\ xy^2 = \lambda \\ x + y + z = 32 \end{cases}$$

setting λ's equal: $y^2z = 2xyz$

setting λ's equal: $2xyz = xy^2$

setting $2xyz$'s equal: $y^2z = xy^2$

Divide by $y^2 \Rightarrow z = x$

(this assumes $y \neq 0$)

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Example 3: Find positive numbers x and y that minimize $f(x,y) = 3x + y + 10$ subject to the constraint $x^2y = 6$.

$\nabla f(x,y) = \langle 3, 1 \rangle$

$g(x,y) = x^2y$

$\nabla g(x,y) = \langle 2xy, x^2 \rangle$

$\nabla f = \lambda \nabla g$

system
$$\begin{cases} 3 = \lambda(2xy) & \Rightarrow \lambda = \dfrac{3}{2xy} \\ 1 = \lambda x^2 & \Rightarrow \lambda = \dfrac{1}{x^2} \\ x^2y = 6 \end{cases}$$

set λ's equal: $\dfrac{3}{2xy} = \dfrac{1}{x^2}$ Note: $x \neq 0, y \neq 0$

$3x^2 = 2xy$

$3x^2 - 2xy = 0$

$x(3x - 2y) = 0$

$x = 0$ Throw out.

If $x = 0$, then $3 = \lambda(2xy)$ and $1 = \lambda x^2$ are impossible

$3x - 2y = 0$

$3x = 2y$

$\frac{3}{2}x = y$

Put $y = \frac{3}{2}x$ into

$x^2y = 6$

$x^2\left(\frac{3}{2}x\right) = 6$

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Ex 2 cont'd:

We know $x = z$:

$$x+y+z=32$$
$$z+y+z=32$$
$$y+2z=32$$

Also know $y^2z = 2xyz$

Divide by yz: $\quad y = 2x$
(this assumes $y\neq 0, z\neq 0$)

Also know $2xyz = xy^2$

Divide by xy: $\quad 2z = y$. Put this into $y+2z=32$
(this assumes $x\neq 0, y\neq 0$)

$$2z+2z=32$$
$$4z=32$$
$$z=8$$
$$y=2z=2(8)=16$$
$$y=2x \Rightarrow x=\frac{y}{2}=\frac{16}{2}=8$$

So, the Maximum P occurs when $x=8$, $y=16$, $z=8$.

$$P=xy^2z = 8(16)^2(8) = 16384$$

$$\boxed{P(8,16,8)=16384}$$

Ex 3 cont'd:

$$x^2\left(\frac{3}{2}x\right)=6$$
$$\frac{3x^3}{2}=6$$
$$3x^3=12$$
$$x^3=4$$
$$x=\sqrt[3]{4}$$
$$y=\frac{3}{2}x = \frac{3}{2}\left(\sqrt[3]{4}\right) = \frac{3\sqrt[3]{4}}{2}$$

The numbers are $x=\sqrt[3]{4}$, $y=\frac{3\sqrt[3]{4}}{2}$

Example 4: A rectangular box without a lid is to be made from 12 square meters of cardboard. Find the maximum volume of such a box.

Due to a power outage, we finished this section on the whiteboard (instead of the computer). See Summer 2015 notes.

Example 5: Find the maximum volume of a rectangular box inscribed in the ellipsoid $x^2 + 3y^2 + 4z^2 = 12$.

Example 6: Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

Example 7: Find the extreme values of $f(x, y, z) = x^2 y^2 z^2$ on the sphere $x^2 + y^2 + z^2 = 1$.

Optimization problems with two constraints:

Suppose we want to find the extrema of $f(x, y, z)$ subject to <u>two</u> constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$. Then, the gradient of *f* must be a linear combination of the gradients of *g* and *h*. In other words, if *f* has an extremum at (x_0, y_0, z_0), then

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$, where λ and μ are scalars.

Lagrange's method results in five equations in five variables:

$$
\begin{aligned}
f_x &= \lambda g_x + \mu h_x \\
f_y &= \lambda g_y + \mu h_y \\
f_z &= \lambda g_z + \mu h_z \\
g(x, y, z) &= k \\
h(x, y, z) &= c
\end{aligned}
$$

<u>**Example 8:**</u> Find the extreme values of $f(x, y, z) = 3x - y - 3z$ on the curve of intersection of $x + y - z = 0$ and $x^2 + 2z^2 = 1$.

Example 9: Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2z = 6$ and $x + y = 12$.

Example 10: Suppose the temperature at each point on the sphere $x^2 + y^2 + z^2 = 50$ is given by the function $T(x, y, z) = 100 + x^2 + y^2$. Find the maximum temperature on the curve formed by the intersection of the sphere and the plane $x - z = 0$.