## 13.1: Introduction to Functions of Several Variables

You are already acquainted with functions of several variables, even if you haven't written them in function notation:

Volume of a rectangular solid: $V(l,w,h) = lwh$

Volume of a right circular cone: $V(r,h) = \frac{1}{3}\pi r^2 h$

**Example 1:** Suppose $z = f(x,y) = x^3 + 2xy + 3y^2$. Evaluate $f(-2,3)$.

$$f(-2,3) = (-2)^3 + 2(-2)(3) + 3(3)^2$$
$$= -8 - 12 + 27$$
$$= \boxed{7}$$

**Example 2:** Suppose $g(x,y,z) = 2xz^2 - 3y^3 + 5y^2 z$. Evaluate $g(3,4,-2)$.

$$g(3,4,-2) = 2(3)(-2)^2 - 3(4)^3 + 5(4)^2(-2)$$
$$= 24 - 3(64) - 10(16) = 24 - 192 - 160$$
$$= \boxed{-328}$$

**Domain and range of functions of several variables:**

The *domain* of a function of *n* variables is the set of points (inputs) in $\mathbb{R}^n$ for which the function results in a valid output. The *range* of a function is the set of all outputs of the function.

Note: The graph of a function of *n* variables is a set of points in $\mathbb{R}^{n+1}$. (When we combine the output of the function with the values of all the input variables, we add a dimension.)

For example, the graph of a function of one variable is a curve in $\mathbb{R}^2$. If we start with $f(x)$, we can let $y = f(x)$, and then the graph consists of ordered pairs $(x, y)$.

Similarly, the graph of a function of two variables is a ~~curve~~ surface in $\mathbb{R}^3$. If we start with $f(x,y)$, we can let $z = f(x,y)$, and then the graph consists of ordered triples $(x, y, z)$.

~~To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.~~

**Example 3:** Find the domain ~~and range~~ of $f(x,y) = \frac{xy}{x-y}$.

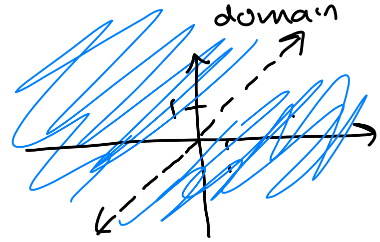
Cannot have a zero denominator.

so $x - y \neq 0$

$x \neq y$

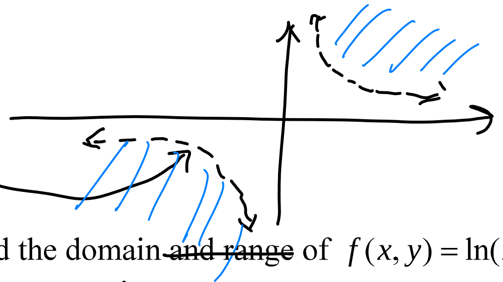
Domain: $\{(x,y) \mid x \neq y\}$

Sketch the domain



(Domain is all pairs not on this line)

$y = \frac{6}{x}$



**Example 4:** Find the domain ~~and range~~ of $f(x, y) = \ln(xy - 6)$.

Can only apply ln function to numbers $> 0$

so, $xy - 6 > 0$ → if $x > 0$, then $y > \frac{6}{x}$

$xy > 6$ → if $x < 0$, then $y < \frac{6}{x}$

Note: $x = 0$ is not in domain

Domain: $\boxed{\{(x,y) \mid xy > 6\}}$

~~**Example 5:** Find the domain and range of $f(x, y) = \ln(xy - 6)$.~~

**Example 6:** Find the domain and range of $f(x, y) = \sqrt{4 - x^2 - 9y^2}$.

$4 - x^2 - 9y^2 \geq 0$

$4 \geq x^2 + 9y^2$

$x^2 + 9y^2 \leq 4$

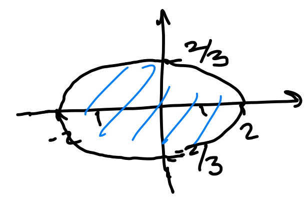
$\frac{x^2}{4} + \frac{9y^2}{4} \leq 1$

$\frac{x^2}{4} + \frac{y^2}{(4/9)} \leq 1$

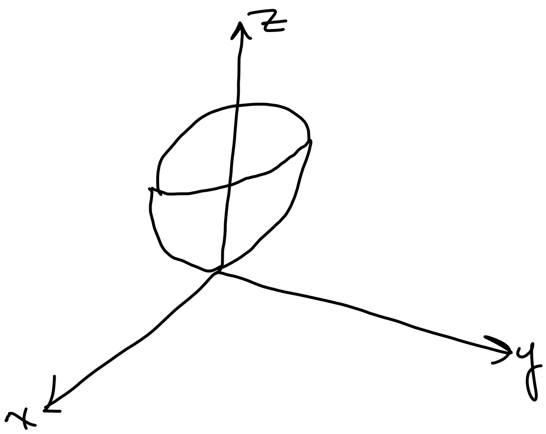
Domain: All points inside the ellipse $\frac{x^2}{4} + \frac{y^2}{4/9} = 1$

Range: $[0, 2]$ ← The max value of $f(x,y)$ occurs when $x=0, y=0$

$F(0,0) = \sqrt{4-0-0} = 2$




**Example 7:** Sketch the graph of the function $f(x, y) = \frac{x^2}{9} + \frac{y^2}{16}$.

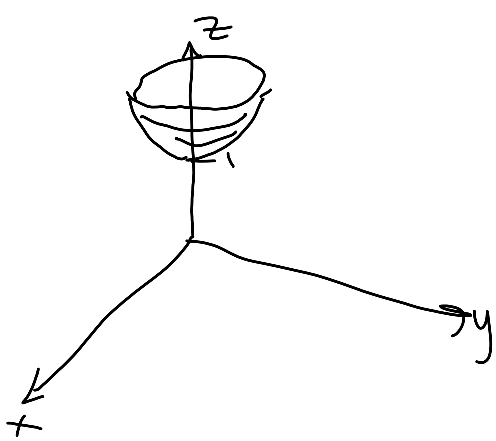



$z = \frac{x^2}{9} + \frac{y^2}{16}$

Range: $[0, \infty)$

Domain: $\mathbb{R}^2$

**Example 8:** Sketch the graph of the function $f(x, y) = \sqrt{1 + x^2 + y^2}$.




$z = f(x,y) \Rightarrow z = \sqrt{1 + x^2 + y^2}$

$z^2 = 1 + x^2 + y^2$

$z^2 - x^2 - y^2 = 1$

$1 + x^2 + y^2 \geq 0$ always true

Domain: $\mathbb{R}^2$

Range: $[1, \infty)$

**Level curves:**

A *level curve* for a function of two variables is a set of points in $\mathbb{R}^2$ for which the function value (output) is constant.

For example, if $z = f(x, y)$, then the level curve for $z = 1$ is the set of points $(x, y)$ for which $z = f(x, y) = 1$. Setting $z = 1$ in the equation of the function produces an equation in *x* and *y* only. The graph of this equation is the level curve for $z = 1$. If we draw the level curves for $z = 1$, $z = 2$, $z = 3$, etc. in the *xy*-plane, they'll help us visualize the graph of the function. A drawing of level curves is called a *contour map*.

Recall: The intersection of a surface in $\mathbb{R}^3$ with a plane is called the *trace* of that surface in the plane. So, for a curve in which $z = f(x, y)$, the level curve for $z = c$ is just the trace of the surface in the plane $z = c$.

Note: In order for a contour map to be helpful in visualization, the *z*-values for the level curves should be equally spaced. Then, level curves that are far apart indicate than *z* is changing slowly. Level curves very close together indicate a rapid change in *z*.

$f(x,y) = \sqrt{16-x^2-y^2}$

$k = \sqrt{16-x^2-y^2}$

$k^2 = 16-x^2-y^2$

$x^2+y^2 = 16-k^2$

**Example 9:** Draw several level curves for $\cancel{f(x, y) = x^2 - y^2}$.

$z = \sqrt{16-x^2-y^2}$

Choose values for $z$:

$z=0 \Rightarrow 0 = \sqrt{16-x^2-y^2}$

$0 = 16-x^2-y^2$

$x^2+y^2=16$

$z=1 \Rightarrow k=1 \Rightarrow x^2+y^2 = 16-1^2$

$x^2+y^2=15$

Circle radius $\sqrt{15}$

$z=2 \Rightarrow k=2 \Rightarrow x^2+y^2=16-2^2$

$x^2+y^2=12$

Radius $= \sqrt{12}$

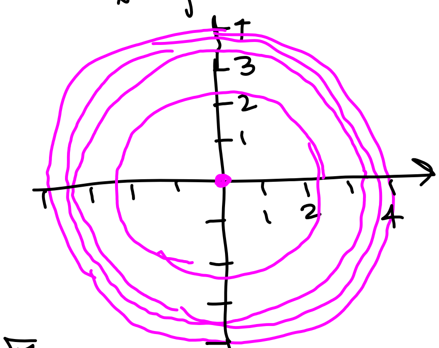
$z=3 \Rightarrow x^2+y^2 = 7$

Radius $= \sqrt{7}$

Circles of radii $4, \sqrt{15}, \sqrt{12}, \sqrt{7}$

$\approx 4, 3.87, 3.46, 2.65$

$z = 4 \Rightarrow$ Radius $= 0$




**Example 10:** Draw several level curves for $f(x, y) = x - y^2$.

**Level surfaces:**

When we extend the notion of level curves to functions of three variables, we get *level surfaces*. A level surface for the function $f(x, y, z)$ is the set of points in $\mathbb{R}^3$ for which $f(x, y, z) = k$ ($k$ a constant).

<u>**Example 11:**</u> Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces look like?

They are spheres - see summer notes.