13.3
~~13.5:~~ Partial Derivatives

Consider a function of two or more variables, such as $f(x,y)=x^3+2x^3y^3-y^5$. What would the derivative represent? Rate of change with respect to what?

Partial differentiation is the process of finding the rate of change in a function with respect to one variable, while holding the other variables constant.

Definition: Suppose $z=f(x,y)$ is a function of *x* and *y*.

The partial derivative of *f* (or *z*) with respect to *x* is

$f_x(x,y) = \frac{\partial z}{\partial x}$

$$f_x(x,y)=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x},\text{ provided this limit exists.}$$

The partial derivative of *f* (or *z*) with respect to *y* is

$$f_y(x,y)=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y},\text{ provided this limit exists.}$$

The partial derivative with respect to *x*, $f_x(x,y)$, gives the slope of the surface in the direction of the *x*-axis.

The partial derivative with respect to *y*, $f_y(x,y)$, gives the slope of the surface in the direction of the *y*-axis.

(We often refer to these as the first partial derivatives, to distinguish them from the second and higher-order partial derivatives.)

Example 1: Find the first partial derivatives of $f(x,y)=x^3+2x^3y^3-y^5$.

$f(x,y)=x^3+2x^3y^3-y^5$

$f_x(x,y)=3x^2+2y^3(3x^2)-0$ [Treat y as constant]

$=\boxed{3x^2+6x^2y^3}$

$f_y(x,y)=0+2x^3(3y^2)-5y^4$ [Treat x as constant]

$=\boxed{6x^3y^2-5y^4}$

Example 2: Suppose $z = 9 - x^2 - y^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\sqrt{3}, 5)$.

$$\frac{\partial z}{\partial x} = 0 - 2x - 0 = -2x$$

$$\frac{\partial z}{\partial y} = 0 - 0 - 2y = -2y$$

$$\left.\frac{\partial z}{\partial x}\right|_{(\sqrt{3},5)} = \left.-2x\right|_{(\sqrt{3},5)} = \boxed{-2\sqrt{3}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(\sqrt{3},5)} = \left.-2y\right|_{(\sqrt{3},5)} = -2(5) = \boxed{-10}$$

Example 3: Suppose $z = \ln\sqrt{xy}$. Find all the first partial derivatives.

$$z = \ln(xy)^{1/2} = \tfrac{1}{2}\ln(xy)$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\cdot\frac{1}{xy}\frac{\partial}{\partial x}(xy) \leftarrow \text{chain rule}$$

$$= \frac{1}{2}\cdot\frac{1}{xy}(y) = \boxed{\frac{1}{2x}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\cdot\frac{1}{xy}\frac{\partial}{\partial y}(xy)$$

$$= \frac{1}{2}\cdot\frac{1}{xy}(x)$$

$$= \boxed{\frac{1}{2y}}$$

Example 4: Suppose $f(x, y) = \cos(x^2 + y^2)$. Find all the first partial derivatives.

$$f_x(x,y) = -\sin(x^2+y^2)\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \left[-\sin(x^2+y^2)\right](2x)$$

$$= \boxed{-2x\sin(x^2+y^2)}$$

$$f_y(x,y) = -\sin(x^2+y^2)\frac{\partial}{\partial y}(x^2+y^2)$$

$$= \left[-\sin(x^2+y^2)\right](0+2y)$$

$$= \boxed{-2y\sin(x^2+y^2)}$$

$$z = x^2(1+xy)^{1/2}$$

Example 5: Suppose $z = x^2\sqrt{1 + xy}$. Find all the first partial derivatives.

$$\frac{\partial z}{\partial x} = x^2\frac{\partial}{\partial x}(1+xy)^{1/2} + (1+xy)^{1/2}\frac{\partial}{\partial x}(x^2)$$

$$= x^2\left(\tfrac{1}{2}\right)(1+xy)^{-1/2}(0+y) + (1+xy)^{1/2}(2x)$$

$$= \frac{x^2y}{2\sqrt{1+xy}} + \frac{2x\sqrt{1+xy}}{1}\left(\frac{2\sqrt{1+xy}}{2\sqrt{1+xy}}\right)$$

$$\frac{x^2y + 4x(1+xy)}{2\sqrt{1+xy}}$$

$$= \frac{x^2y + 4x + 4x^2y}{2\sqrt{1+xy}} = \boxed{\frac{5x^2y+4x}{2\sqrt{1+xy}}}$$

$$\frac{\partial z}{\partial y} = x^2\left(\tfrac{1}{2}\right)(1+xy)^{-1/2}(0+x)$$

$$= \boxed{\frac{x^3}{2\sqrt{1+xy}}}$$

Example 6: Suppose $f(x, y) = \frac{xy}{x^2 + y^2}$. Find all the first partial derivatives.

Quotient Rule: $$f_x(x,y) = \frac{(x^2+y^2)\frac{\partial}{\partial x}(xy) - xy\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(y) - xy(2x+0)}{(x^2+y^2)^2} = \frac{x^2y + y^3 - 2x^2y}{(x^2+y^2)^2}$$

$$= \boxed{\frac{y^3 - x^2y}{(x^2+y^2)^2}}$$

From symmetry of variables:

$$\boxed{f_y(x,y) = \frac{x^3 - y^2x}{(x^2+y^2)^2}}$$

Example 7: Suppose $f(x,y,z) = 3xyz^2 + \frac{1}{xy} - 2z$. Find all the first partial derivatives.

See Summer 2015 notes

Example 8: Suppose $z = x^2 e^{xy^2}$. Find all the first partial derivatives.

$$z = x^2 e^{xy^2}$$

$$\frac{\partial z}{\partial x} = x^2 \frac{\partial}{\partial x}\left(e^{xy^2}\right) + e^{xy^2}\frac{\partial}{\partial x}\left(x^2\right)$$

$$= x^2 e^{xy^2}(y^2) + e^{xy^2}(2x)$$

$$= \boxed{x^2y^2e^{xy^2} + 2xe^{xy^2}}$$

$$\frac{\partial z}{\partial y} = x^2 e^{xy^2}(2xy)$$

$$= \boxed{2x^3y\,e^{xy^2}}$$

Example 9: Suppose $f(x,y,z) = x^2y^3 + 2xyz - 3yz$. Find all the first partial derivatives at the point $(-2,1,2)$.

See Summer 2015 notes

Example 10: Find the slope in the x- and y-directions of the surface given by $f(x,y) = x\sin(x+y)$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{3}\right)$.

See summer 2015 notes

Higher-order partial derivatives:

The second partial derivatives of $z = f(x,y)$ are defined as follows: w.r.t.

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$ Differentiate 1st w/respect to x, then w.r.t. x

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$ Differentiate 1st w.r.t. y then w.r.t y

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$ Differentiate 1st w.r.t x then w.r.t y

$$\cancel{f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}}$$

mixed 2nd partial derivatives

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$ 1st w.r.t. y then w.r.t. x

Example 11: Suppose $f(x,y) = x\sin(4x-3y)$. Find all the second partial derivatives.

See summer 2015 notes

Example 12: Suppose $f(x,y,z) = \dfrac{3z^2}{x+2y}$. Find all the second partial derivatives.

$$f(x,y,z) = 3z^2(x+2y)^{-1}$$

$$f_x(x,y,z) = 3z^2(-1)(x+2y)^{-2}\frac{\partial}{\partial x}(x+2y)$$

$$= -3z^2(x+2y)^{-2}(1+0) = -3z^2(x+2y)^{-2}$$

$$f_y(x,y,z) = 3z^2(-1)(x+2y)^{-2}(0+2) = -6z^2(x+2y)^{-2}$$

$$f_z(x,y,z) = 3(x+2y)^{-1}(2z) = 6z(x+2y)^{-1}$$

$$f_{xz}(x,y,z) = \frac{\partial}{\partial z}\left(-3z^2(x+2y)^{-2}\right)$$

$$= -3(x+2y)^{-2}(2z)$$

$$= \boxed{-6z(x+2y)^{-2}}$$

See next page

$$f_{yz}(x,y,z) = \frac{\partial}{z}\left(-6z^2(x+2y)^{-2}\right)$$

$$= -6(x+2y)^{-2}(2z)$$

$$= \boxed{-12z(x+2y)^{-2}}$$

$$f_{zy}(x,y,z) = \frac{\partial}{\partial y}\left(6z(x+2y)^{-1}\right)$$

$$= 6z(-1)(x+2y)^{-2}(0+2)$$

$$= \boxed{-12z(x+2y)^{-2}}$$

Note: these are equal!

Ex 12 cont'd:

$$f_{zx}(x,y) = \frac{\partial}{\partial x}\left[6z(x+2y)^{-1}\right] = 6z(-1)(x+2y)^{-2}(1+0)$$
$$= \boxed{-6z(x+2y)^{-2}}$$

Note: same as f_{xz} on previous page

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(x,y)\right] = \frac{\partial}{\partial y}\left[-3z^2(x+2y)^{-2}\right]$$
$$= -3z^2(-2)(x+2y)^{-3}(0+2)$$
$$= \boxed{12z^2(x+2y)^{-3}}$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left[f_y(x,y)\right] = \frac{\partial}{\partial x}\left[-6z^2(x+2y)^{-2}\right] = -6z^2(-2)(x+2y)^{-3}(1+0)$$
$$= \boxed{12z^2(x+2y)^{-3}}$$

Note: these are equal!

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(x,y)\right] = \frac{\partial}{\partial x}\left[-3z^2(x+2y)^{-2}\right]$$
$$= -3z^2(-2)(x+2y)^{-3}(1+0) = \boxed{6z^2(x+2y)^{-3}}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(x,y)\right] = \frac{\partial}{\partial y}\left[-6z^2(x+2y)^{-2}\right] = -6z^2(-2)(x+2y)^{-3}(0+2)$$
$$= \boxed{24z^2(x+2y)^{-3}}$$

$$f_{zz}(x,y) = \frac{\partial}{\partial z}\left[f_z(x,y)\right] = \frac{\partial}{\partial z}\left[6z(x+2y)^{-1}\right]$$
$$= \boxed{6(x+2y)^{-1}}$$

> Theorem: Equality of Mixed Partial Derivatives
> (sometimes known as Clairut's Theorem).
>
> If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then
>
> $$f_{xy}(x, y) = f_{yx}(x, y) \text{ for every } (x, y) \text{ in } R.$$

This theorem also applies to third- and higher order derivatives, and to functions of three or more variables. As long as all the higher-order partial derivatives are continuous, all the mixed partial derivatives of that order will be equal.

Example 13: Su~~ppose $w(x, y, z) = e^{x^2yz}$. Find~~ $w_{xyz}(x, y, z)$, $w_{zyx}(x, y, z)$, $w_{xyx}(x, y, z)$, and $w_{xxy}(x, y, z)$. See a different example in summer notes.

Example 14: Determine whether the following functions satisfy the partial differential equation (PDE) $u_{xx} + u_{yy} = 0$, known as Laplace's equation.

a) $u = x^3 + 3xy^2$

b) $u = e^{-x}\cos y - e^{-y}\cos x$

a
$u_x = 3x^2 + 3y^2$
$u_y = 0 + 3x(2y) = 6xy$
$u_{xx} = 6x + 0 = 6x$
$u_{yy} = 6x$
Is $u_{xx} + u_{yy} = 0$ true?
$u_{xx} + u_{yy} = 6x + 6x = 12x \neq 0$

No, does not satisfy the PDE

b
$u_x = (e^{-x}\cos y)(-1) - e^{-y}(-\sin x)$
$= -e^{-x}\cos y + e^{-y}\sin x$
$u_y = e^{-x}(-\sin y) - e^{-y}(-1)\cos x$
$= -e^{-x}\sin y + e^{-y}\cos x$
$u_{xx} = -e^{-x}(-1)\cos y + e^{-y}\cos x$
$= e^{-x}\cos y + e^{-y}\cos x$

See next page

Ex 14 cont'd

$$u_{yy} = -e^{-x}\cos y + e^{-y}(-1)\cos x$$
$$= -e^{-x}\cos y - e^{-y}\cos x$$

Is $u_{xx} + u_{yy} = 0$ true?

$$u_{xx} + \underline{u_{yy}} = \cancel{e^{-x}\cos y} + \cancel{e^{-y}\cos x} - \cancel{e^{-x}}\cos y - e^{-y}\cancel{\cos x} = 0$$

$\boxed{\text{Yes,}}$ the PDE is true.

$u = e^{-x}\cos y - e^{-y}\cos x$ satisfies Laplace's equation.