

13.4: Differentials

Essential question in calculus: If we make an incremental change in one variable, what happens to the other variables?

Increments: Δx , Δy , Δz , etc.

Differentials: dx , dy , dz , etc.

For $y = f(x)$, $\Delta y = f(x + \Delta x) - f(x)$

For $z = f(x, y)$, $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$.

For the independent variable(s), we define the differential to be equal to the increment. For the dependent variable, we define the differential in terms of the derivative.

For $y = f(x)$, we let $dx = \Delta x$. Then the differential of y is $dy = f'(x) dx$.

The differential dy has two uses:

- (a) as an approximation for the increment Δy . ($dy \approx \Delta y$)
- (b) as a building block for calculus concepts (integration, etc).

To extend the concept of differentials to functions of two or more variables, we need to use partial derivatives.

Definition:

If $z = f(x, y)$, then the differentials of the independent variables x and y are the increments:

$$dx = \Delta x \qquad dy = \Delta y$$

The total differential of the dependent variable z is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x, y) dx + f_y(x, y) dy.$$

Note: For small increments of z , $dz \approx \Delta z$ (for differentiable functions).

Example 1: Find the total differential for $z = x^2y + x^4 - 5y^3$.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = (2xy + 4x^3) dx + (x^2 - 15y^2) dy$$

$$\left\{ \begin{array}{l} \frac{\partial z}{\partial x} = y(2x) + 4x^3 + 0 \\ \qquad \qquad = 2xy + 4x^3 \\ \frac{\partial z}{\partial y} = x^2 + 0 - 15y^2 \\ \qquad \qquad = x^2 - 15y^2 \end{array} \right.$$

Example 2: Find the total differential for $w = x^2 y z^2 + \sin(yz)$. $\frac{\partial w}{\partial x} = 2xy z^2 + 0 = 2xy z^2$

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

$$\frac{\partial w}{\partial y} = x^2 z^2 + [\cos(yz)](z) = x^2 z^2 + z \cos(yz)$$

$$\frac{\partial w}{\partial z} = x^2 y (2z) + [\cos(yz)](y) = 2x^2 y z + y \cos(yz)$$

$$dw = 2xy z^2 dx + (x^2 z^2 + z \cos(yz)) dy + (2x^2 y z + y \cos(yz)) dz$$

Differentiability:

Definition: A function f given by $z = f(x, y)$ is *differentiable* at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both $\varepsilon_1 \rightarrow 0$ and $\varepsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

The function f is differentiable in a region R if it is differentiable at each point in R .

Important Note: Existence of the partial derivatives f_x and f_y at the point (x_0, y_0) does NOT guarantee the function is differentiable at (x_0, y_0) .

However, if the partial derivatives are continuous in a ^{open} region, then the function is differentiable in the region:

Theorem: If f is a function of x and y , and if f_x and f_y are continuous on an open region R , then f is differentiable on R .

Using the total differential as an approximation for the increment:

Example 3: Suppose $z = f(x, y) = \frac{y}{x}$. Calculate the increment Δz as y changes from 2 to 2.1, and x changes from 1 to 1.03. Use the total differential dz as an approximation for Δz , and compare it to the actual value of Δz .

$$z = yx^{-1}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= -yx^{-2} dx + x^{-1} dy$$

$$= -\frac{y}{x^2} dx + \frac{1}{x} dy$$

$$dz = -\frac{2}{1^2} (0.03) + \frac{1}{1} (0.1)$$

$$= -0.06 + 0.10$$

$$= \boxed{0.04}$$

$dy = \Delta y = 2.1 - 2 = 0.1$
 $dx = \Delta x = 1.03 - 1 = 0.03$

$x=1$
 $y=2$
 $dx=0.03$
 $dy=0.1$

Error analysis:

When using a formula to compute a physical quantity, any measurement error in the independent variables will cause propogated error in the dependent variable. We can use the total differential to estimate the propogated error.

Estimating propogated error:

Suppose that x and y are the measured values of two variables, and that $dx = \Delta x$ and $dy = \Delta y$ are the maximum possible measurement errors. Then the estimated propogated error in $z = f(x, y)$ is

$$\text{Estimated propogated error: } dz = f_x(x, y) dx + f_y(x, y) dy$$

$$\text{Estimated relative error: } \frac{dz}{z}$$

Example 4: The measurements of the height and inside radius of a right circular cone are 50 feet and 30 feet, respectively. The maximum possible error in each measurement is about 3 inches per each 10 feet of measured length. Approximate the maximum propogated error and the relative error in computing the volume of the cone.

$V(r, h) = \frac{1}{3} \pi r^2 h$
 Find dr and dh : (error)
 radius = $r = 30' \Rightarrow dr = 9 \text{ in} \left(\frac{1 \text{ ft}}{12 \text{ in}} \right) = 0.75 \text{ ft}$
 height = $h = 50' \Rightarrow dh = 15 \text{ in} \left(\frac{1 \text{ ft}}{12 \text{ in}} \right) = 1.25 \text{ ft}$
 $dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$
 $= \frac{2}{3} \pi r h dr + \frac{1}{3} \pi r^2 dh$

$dV = \frac{2}{3} \pi (30 \text{ ft})(50 \text{ ft})(0.75 \text{ ft}) + \frac{1}{3} \pi (30 \text{ ft})^2 (1.25 \text{ ft})$
 $= 750 \pi \text{ ft}^3 + 375 \pi \text{ ft}^3$
 $= 1125 \pi \text{ ft}^3$
 Maximum propogated error

Relative error = $\frac{dV}{V} = \frac{1125 \pi \text{ ft}^3}{\frac{1}{3} \pi (30 \text{ ft})^2 (50 \text{ ft})} = 0.075$
 $\Rightarrow 7.5\%$