13.5: Chain Rules for Functions of Several Variables

Chain Rule (One Independent Variable):

Suppose $w = f(x, y)$, where f is a differentiable function of x and y. Also suppose $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Then, w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

multiplying both sides by dt results in:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy$$

(the total differential)

Example 1: Suppose that $w = \cos(x - y)$, $x = t^2$, $y = 1$. Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$= [-\sin(x-y)](1)(2t) + (-\sin(x-y))(-1)(0)$$

$$= -2t\sin(x-y)$$

$$= \boxed{-2t\sin(t^2-1)} \leftarrow x = t^2, y = 1$$

$$\frac{\partial w}{\partial x} = [-\sin(x-y)](1) = -\sin(x-y)$$

$$\frac{\partial w}{\partial y} = [-\sin(x-y)](-1) = \sin(x-y)$$

OR

$$w = \cos(t^2 - 1)$$

$$\frac{dw}{dt} = [-\sin(t^2-1)](2t) = \boxed{-2t\sin(t^2-1)}$$

OR $$\frac{dw}{dt} = [-\sin(x-y)]\left(1\frac{dx}{dt} - 1\frac{dy}{dt}\right) = [-\sin(x-y)](2t - 0)$$ ✓ ok

Example 2: Suppose that $w = xy\cos(z)$, $x = t$, $y = t^2$, $z = \arccos(t)$. Find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= (y\cos(z))(1) + (x\cos(z))(2t) + xy(-\sin z)\left(-\frac{1}{\sqrt{1-t^2}}\right)$$

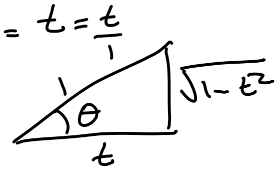
$$= t^2\cos(\arccos t) + t\cos(\arccos t)(2t) + t(t^2)(-\sin(\arccos t))\left(-\frac{1}{\sqrt{1-t^2}}\right)$$

$$= t^2(t) + t(t)(2t) + \frac{t^3}{\sqrt{1-t^2}}\sin(\arccos t)$$

$$= t^3 + 2t^3 + \frac{t^3}{\cancel{\sqrt{1-t^2}}}\cdot\cancel{\sqrt{1-t^2}} = t^3 + 2t^3 + t^3 = \boxed{4t^3}$$

$\theta = \arccos(t)$

$\cos\theta = t = \frac{t}{1}$



$\sin(\arccos t) = \sin\theta = \frac{\sqrt{1-t^2}}{1}$

Ex $1\frac{1}{2}$: Suppose $w = \underbrace{x}_{1st}\underbrace{\sin y}_{2nd}$, $x = t-1$, $y = t^2$. Find $\frac{dy}{dt}$.

$$\frac{dw}{dt} = x\frac{d}{dt}(\sin y) + (\sin y)\frac{d}{dt}(x) \qquad \longleftarrow \text{product rule}$$

$$= x(\cos y)\frac{dy}{dt} + (\sin y)\frac{dx}{dt}$$

$$= (x\cos y)(2t) + (\sin y)(1)$$

$$= (t-1)(\cos(t^2))(2t) + \sin(t^2) = \boxed{(2t^2-2t)\cos(t^2) + \sin(t^2)}$$

Using chain rule from previous page: $w = x\sin y$, $x = t-1$, $y = t^2$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (\sin y)(1) + (x\cos y)(2t)$$

$$= \boxed{\sin(t^2) + 2t(t-1)\cos(t^2)} \qquad \text{same as before}$$

Ex $2\frac{1}{2}$: $w = xy\cos z$, $x = \ln(t)$, $y = \sqrt{t^2+1} = (t^2+1)^{1/2}$, $z = t^2$

Putting t's in: $w(t) = \ln(t)\sqrt{t^2+1}\,\cos(t^2)$

3-factor product rule

Use chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y\cos z)\left(\frac{1}{t}\right) + (x\cos z)\left(\frac{1}{2}\right)(t^2+1)^{-1/2}(2t) + xy(-\sin z)(2t)$$

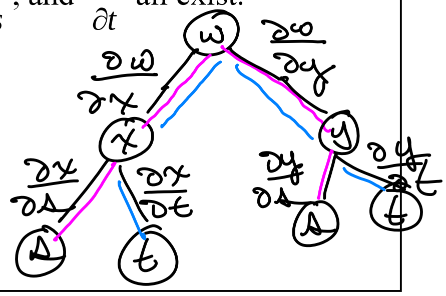
Plug in t's

<u>Chain Rule (Two Independent Variables)</u>:

Suppose $w = f(x, y)$, where f is a differentiable function of x and y. Also suppose $x = g(s,t)$ and $y = h(s,t)$, such that the first partial derivatives $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial y}{\partial t}$ all exist.

Then, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ both exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$



<u>Example 3:</u> Suppose that $w = \sin(2x+3y)$, $x = s+t$, $y = s-t$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$ at the point where $s = 0$, $t = \frac{\pi}{4}$.

$$\frac{\partial w}{\partial x} = (\cos(2x+3y))(2) = 2\cos(2x+3y)$$

$$\frac{\partial w}{\partial y} = (\cos(2x+3y))(3) = 3\cos(2x+3y)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} = 2\cos(2x+3y)(1) + 3\cos(2x+3y)(-1)$$

$$= 2\cos(2x+3y) - 3\cos(2x+3y) = -\cos(2x+3y)$$

$$= -\cos(2(s+t)+3(s-t)) = -\cos(5s-t)$$

$$\left.\frac{\partial w}{\partial t}\right|_{\substack{s=0\\ t=\frac{\pi}{4}}} = -\cos\left(5(0)-\frac{\pi}{4}\right) = -\cos\left(-\frac{\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

Repeat for $\frac{\partial w}{\partial s}$ (see summer notes)

<u>Example 4:</u> Suppose that $w = \sqrt{25-5x^2-5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

<u>Example 5:</u> Suppose that $w = x^2 - 2xy + y^2$, $x = r+\theta$, $y = r-\theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$.

Chain Rule (Implicit Differentiation):

Suppose the equation $F(x,y)=0$ defines y implicitly as a differentiable function of x.
Then,

$$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)}, \text{ provided } F_y(x,y)\neq 0.$$

$F_y\,dy=-F_x\,dx$

$F_x\,dx+F_y\,dy=0$

See next Page

Suppose the equation $F(x,y,z)=0$ defines z implicitly as a differentiable function of x and y.

Then,

$$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}, \text{ provided } F_z(x,y,z)\neq 0.$$

Example 6: Suppose that $2x^2-3y^3=xy^2-5$. Find $\frac{dy}{dx}$.

Get 1 side 0: $2x^2-3y^3-xy^2+5=0$

Implicit diff (Calc I): $\frac{d}{dx}(2x^2-3y^3-xy^2+5)=\frac{d}{dx}(0)$

$4x-9y^2\frac{dy}{dx}-\left[x\,2y\frac{dy}{dx}+y^2(1)\right]+0=0$

$4x-9y^2\frac{dy}{dx}-2xy\frac{dy}{dx}-y^2=0$

$4x-y^2=9y^2\frac{dy}{dx}+2xy\frac{dy}{dx}$

$4x-y^2=(9y^2+2xy)\frac{dy}{dx}$

$\boxed{\frac{4x-y^2}{9y^2+2xy}=\frac{dy}{dx}}$

Example 7: Suppose that $\ln(x^2+y^2)+2xy=8$. Find $\frac{dy}{dx}$.

Ex 6 Using Calc III Chain Rule

$F(x,y)=2x^2-3y^3-xy^2+5$

$F_x(x,y)=4x-0-y^2+0=4x-y^2$

$F_y(x,y)=0-9y^2-x(2y)+0$

$=-9y^2-2xy$

$\frac{dy}{dx}=-\frac{F_x(x,y)}{F_y(x,y)}$

$=-\frac{4x-y^2}{-9y^2-2xy}$

$=\boxed{\frac{4x-y^2}{9y^2+2xy}}$

same

Example 8: Suppose that $z^3+2xy^2=\ln z+x^3y^4z^2$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F(x,y,z)=z^3+2xy^2-\ln z-x^3y^4z^2=0$

$F_x(x,y,z)=0+2y^2-0-3x^2y^4z^2=2y^2-3x^2y^4z^2$

$F_y(x,y,z)=0+2x(2y)-0-x^3z^2(4y^3)=4xy-4x^3y^3z^2$

$F_z(x,y,z)=3z^2+0-\frac{1}{z}-x^3y^4(2z)=3z^2-\frac{1}{z}-2x^3y^4z$

$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)}=\boxed{-\frac{2y^2-3x^2y^4z^2}{3z^2-\frac{1}{z}-2x^3y^4z}}$

See next page

Ex 8: cont'd)

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \boxed{-\frac{4xy-4x^3y^3z^2}{3z^2-\frac{1}{z}-2x^3y^4z}}$$

Example 9: Suppose that $z = e^x \sin(y+z)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

See summer notes for the examples we didn't do in class.

Example 10: Suppose that $yz^4 + x^2y^3 = e^{xyz}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Example 11: Suppose that $x\ln y + y^2z + z^2 = 8$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Why is $\frac{dy}{dx} = - \frac{F_x(x,y)}{F_y(x,y)}$

Suppose $w = F(x,y) = 0$.

$$\frac{\partial}{\partial x}\left(F(x,y)\right) = \frac{\partial}{\partial x}(0)$$

$$\frac{\partial w}{\partial x} \cdot \frac{d}{dx}(x) + \frac{\partial w}{\partial y} \cdot \frac{d}{dx}(y) = 0$$

$$\frac{\partial w}{\partial x}(1) + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$F_x(x,y) + F_y(x,y)\frac{dy}{dx} = 0$$

solve for $\frac{dy}{dx}$: $F_y(x,y)\frac{dy}{dx} = -F_x(x,y)$

$$\frac{dy}{dx} = - \frac{F_x(x,y)}{F_y(x,y)}$$