13.6: Directional Derivatives and Gradients

Given $z = f(x, y)$, the partial derivatives f_x and f_y represent the rates of change of *z* in the *x*- and *y*-directions. That is, these are the rates of change of *z* along the vectors **i** and **j**.

We can calculate the rate of change in *z* along any vector **u**, called the *directional derivative* in the direction of **u**.

Definition: Directional Derivative

Let *f* be a function of two variables *x* and *y*, and let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. Then, the directional derivative of *f* in the direction of **u** is

$$D_{\mathbf{u}} f(x, y) = \lim_{t \to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x, y)}{t},$$

provided this limit exists.

The above limit definition is not very practical for calculating directional derivatives. Instead, we'll use this theorem:

Theorem:

Let $f(x, y)$ be a differentiable function.

Then, the directional derivative of *f* in the direction of $\underbrace{\mathbf{u} = \langle \cos\theta, \sin\theta \rangle}$ is

this forces $\vec{u}$ to be a unit vector

Note: $\|\vec{u}\| = 1$

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$
$$= \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle.$$

Example 1: Suppose $f(x, y) = x^2 - 2xy^2 + 3y^4$. Find the directional derivative in the direction $\theta = \frac{\pi}{3}$ at the point $(1, 2)$.

$\vec{u} = \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle$ Note: $\|\vec{u}\| = 1$

$f_x(x,y) = 2x - 2y^2 + 0 = 2x - 2y^2$

$f_y(x,y) = 0 - 4xy + 12y^3 = -4xy + 12y^3$

At the point $(1,2)$: $f_x(1,2) = 2(1) - 2(2)^2 = 2 - 8 = -6$

$f_y(1,2) = -4(1)(2) + 12(2)^3 = -8 + 96 = 88$

$D_{\vec{u}} f(1,2) = \langle -6, 88 \rangle \cdot \langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \rangle = \langle -6, 88 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle =$

$= -6\left(\frac{1}{2}\right) + 88\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{-3 + 44\sqrt{3}}$

Example 2: Find the directional derivative for $f(x, y) = xy$ in the direction $\mathbf{u} = \left\langle \frac{3}{5}, \frac{2}{5} \right\rangle$ at the point $(2,3)$.

$\vec{u} = \langle \frac{3}{5}, \frac{2}{5} \rangle$

Get a unit vector: $\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle \frac{3}{5}, \frac{2}{5} \rangle}{\sqrt{\frac{9}{25} + \frac{4}{25}}} = \frac{1}{\frac{\sqrt{13}}{5}} \langle \frac{3}{5}, \frac{2}{5} \rangle$

$= \frac{5}{\sqrt{13}} \langle \frac{3}{5}, \frac{2}{5} \rangle = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle$ ← Note: think of this as $\langle \cos\theta, \sin\theta \rangle$ in the definition.

$f_x(x,y) = y \Rightarrow f_x(2,3) = 3$

$f_y(x,y) = x \Rightarrow f_y(2,3) = 2$

$D_{\vec{v}} f(2,3) = \langle 3, 2 \rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \rangle = \frac{9}{\sqrt{13}} + \frac{4}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \boxed{\sqrt{13}}$

Example 3: Find the directional derivative for $f(x, y) = e^x \sin y$ in the direction $\mathbf{v} = \langle -1, 5 \rangle$ at the point $\left(1, \frac{\pi}{2}\right)$.

Find a unit vector in the direction of $\vec{v}$:

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -1, 5 \rangle}{\sqrt{1+25}} = \langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \rangle$

$f_x(x,y) = e^x \sin y$

$f_y(x,y) = e^x \cos y$

$f_x(1, \frac{\pi}{2}) = e \sin\frac{\pi}{2} = e$

$f_y(1, \frac{\pi}{2}) = e \cos\frac{\pi}{2} = 0$

$D_{\vec{v}}(1, \frac{\pi}{2}) = \langle e, 0 \rangle \cdot \langle -\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \rangle$

$= \boxed{-\frac{e}{\sqrt{26}}}$

Example 4: Find the directional derivative for $f(x, y) = x^4 + 5y^2$ at point $P(2,1)$ in the direction toward the point $Q(-4,3)$.

$\vec{PQ} = \langle -6, 2 \rangle$

Find a unit vector:

$\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{\langle -6, 2 \rangle}{\sqrt{36+4}} = \langle \frac{-6}{\sqrt{40}}, \frac{2}{\sqrt{40}} \rangle = \langle \frac{-6}{2\sqrt{10}}, \frac{2}{2\sqrt{10}} \rangle = \langle -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

$f_x(x,y) = 4x^3 \Rightarrow f_x(2,1) = 4(2)^3 = 32$

$f_y(x,y) = 10y \Rightarrow f_y(2,1) = 10(1) = 10$

$D_{\vec{u}} f(2,1) = \langle 32, 10 \rangle \cdot \langle -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle = -\frac{96}{\sqrt{10}} + \frac{10}{\sqrt{10}}$

$= \boxed{-\frac{86}{\sqrt{10}}} = -\frac{86\sqrt{10}}{10} = \boxed{-\frac{43\sqrt{10}}{5}}$

The gradient vector:

<u>Definition</u>: <u>The Gradient</u>

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the gradient of f, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle .$$

del f

Note: $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

<u>Example 5:</u> Find the gradient of $z = \ln(x^2 - y)$ at the point $(2,3)$.

$$\nabla z = \left\langle \frac{1}{x^2-y}(2x), \frac{1}{x^2-y}(-1) \right\rangle = \left\langle \frac{2x}{x^2-y}, -\frac{1}{x^2-y} \right\rangle$$

$$\nabla z \Big|_{x=2, y=3} = \nabla z \Big|_{(2,3)} = \left\langle \frac{2(2)}{2^2-3}, -\frac{1}{2^2-3} \right\rangle = \left\langle \frac{4}{1}, -\frac{1}{1} \right\rangle$$

$$= \boxed{\langle 4, -1 \rangle}$$

<u>Theorem</u>: Dot Product Form of the Directional Derivative

Let $f(x, y)$ be a differentiable function. Then, the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} .$$

<u>Example 6:</u> Find the directional derivative for $f(x, y) = 3x^2 - y^2 + 4$ at point $P(-1, 4)$ in the direction toward the point $Q(3,6)$.

$$\overrightarrow{PQ} = \langle 4, 2 \rangle$$

$$\vec{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle 4, 2 \rangle}{\sqrt{16+4}} = \left\langle \frac{4}{\sqrt{20}}, \frac{2}{\sqrt{20}} \right\rangle = \left\langle \frac{4}{2\sqrt{5}}, \frac{2}{2\sqrt{5}} \right\rangle$$

$$= \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\nabla f(x,y) = \langle 6x, -2y \rangle$$

$$\nabla f(-1,4) = \langle 6(-1), -2(4) \rangle = \langle -6, -8 \rangle$$

$$D_{\vec{u}} f(x,y) = \nabla f(-1,4) \cdot \vec{u}$$

$$= \langle -6, -8 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = -\frac{12}{\sqrt{5}} - \frac{8}{\sqrt{5}} = -\frac{20}{\sqrt{5}}$$

$$= -\frac{20\sqrt{5}}{5} = \boxed{-4\sqrt{5}}$$

Directional Derivative and Gradient for Functions of Three Variables:

Let f be a function of three variables with continuous first partial derivatives.

Then the gradient of f is the vector

$$\cancel{\nabla f(x,y)} = \langle f_x(x,y,z),\, f_y(x,y,z),\, f_z(x,y,z) \rangle .$$

$\nabla f(x,y,z)$

The directional derivative of f in the direction of a unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u} .$$

Example 7: Find the directional derivative for $f(x,y,z) = xy + yz + xz$ at the point $P(1,2,-1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Find a unit vector in the direction of $\vec{v}$: $\vec{u} = \dfrac{\langle 2, 1, -1 \rangle}{\sqrt{4+1+1}}$

$= \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$

$\nabla f(x,y,z) = \langle y+z,\, x+z,\, y+x \rangle$

$\nabla f(1,2,-1) = \langle 2-1,\, 1-1,\, 2+1 \rangle = \langle 1, 0, 3 \rangle$

$D_{\vec{u}} f(1,2,-1) = \langle 1,0,3 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle = \frac{2}{\sqrt{6}} + 0 - \frac{3}{\sqrt{6}} = \boxed{-\frac{1}{\sqrt{6}}}$

Example 8: Find the directional derivative for $f(x,y,z) = e^{2x} \cos y \sin z$ at the point $P\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ in the direction of $\mathbf{v} = \langle 2, -1, 2 \rangle$.

see summer notes

Finding the directions of maximum and minimum increase:

Theorem: Properties of the Gradient

Suppose the function f is differentiable at the point (x, y). Then,

1) If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.

2) The direction of maximum increase of f is given by $\nabla f(x, y)$.
   The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.

(maximum decrease)

3) The direction of minimum increase of f is given by $-\nabla f(x, y)$.
   The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

Note: These properties also hold for the gradient of a three-variable function $f(x, y, z)$.

Why are #2 and #3 true?

If $\nabla f(x,y) \neq \vec{0}$, then let ϕ be the angle between $\nabla f(x,y)$ and unit vector $\vec{u}$.

$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = \|\nabla f(x,y)\| \, \|\vec{u}\| \cos\phi = \|\nabla f(x,y)\| (1) \cos\phi$ (because $\vec{u}$ is a unit vector)

Where is $D_{\vec{u}} f(x,y)$ at its maximum? When $\cos\phi = 1$

Where is $D_{\vec{u}} f(x,y)$ at its minimum? When $\cos\phi = -1$

$\cos\phi = 1 \Rightarrow \phi = 0$. So $\vec{u}$ must be in same direction as $\nabla f(x,y)$.

$\cos\phi = -1 \Rightarrow \phi = \pi$. So $\vec{u}$ must point in opposite direction from $\nabla f(x,y)$

Example 9: Find the maximum value of the directional derivative for $f(x, y) = x \tan y$ at the point $P\left(2, \frac{\pi}{4}\right)$. What is the direction of maximum increase?

$\nabla f(x,y) = \langle \tan y, \, x \sec^2 y \rangle$

$\nabla f(x,y) = \langle \tan \frac{\pi}{4}, \, 2 \sec^2 \frac{\pi}{4} \rangle = \langle 1, 2(\sqrt{2})^2 \rangle = \langle 1, 4 \rangle$

$\left(\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \right.$

$\left. \sec \frac{\pi}{4} = \sqrt{2} \right)$

Maximum value of $D_{\vec{u}} \left(2, \frac{\pi}{4}\right)$ is $\sqrt{1+16} = \boxed{\sqrt{17}}$

Direction of maximum increase is $\langle 1, 4 \rangle$.

Example 10: Find the maximum and minimum rate of increase in the function $f(x, y, z) = xy + yz + xz$ at the point $P(1, 2, -1)$.

See Example 7: $\nabla f(1,2,-1) = \langle 1, 0, 3 \rangle$ (Direction of maximum increase)

Maximum rate of increase: $\|\nabla f(1,2,-1)\| = \sqrt{1+9} = \boxed{\sqrt{10}}$

minimum rate of increase is $-\sqrt{10}$, in the direction $\langle -1, 0, -3 \rangle$.

Finding a vector normal to the level curves of a function:

> Theorem:
>
> Let f be a function differentiable at (x_0, y_0) such that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then, $\nabla f(x_0, y_0)$ is orthogonal to the level curve that passes through the point (x_0, y_0).

Example 11: For the function $f(x, y) = x + y^2$, find a normal vector to the level curve through the point $(1, 3)$.

Draw some level curves for $f(x,y) = x + y^2$

$z = k \Rightarrow \quad k = x + y^2$

$x = k - y^2$

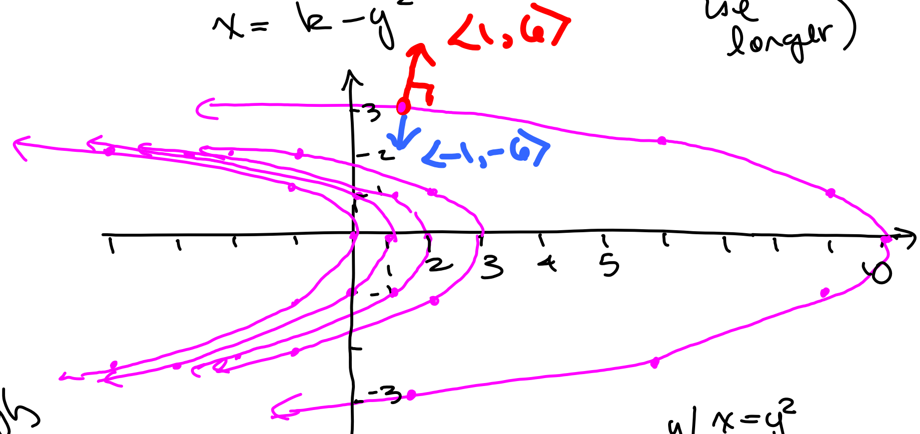
not to scale (length should be longer)

$k = 0 \Rightarrow \quad x = -y^2$

$k = 1 \Rightarrow \quad x = -y^2 + 1$

$k = 2 \Rightarrow \quad x = -y^2 + 2$

$k = 3 \Rightarrow \quad x = -y^2 + 3$



What level curve goes through the point $(1,3)$?

$k = x + y^2$

$k = 1 + 3^2 = 10$ (Want level curve for $k = 10$)

y	$x = y^2$
1	1
2	4
3	9
4	16

$\nabla f(x,y) = \langle 1, 2y \rangle$

$\nabla f(1,3) = \langle 1, 2(3) \rangle = \langle 1, 6 \rangle$

$\boxed{\text{A Normal vector is } \vec{N} = \langle 1, 6 \rangle}$

Another normal vector is $\boxed{\vec{N}_2 = \langle -1, -6 \rangle}$