13.7: Tangent Planes and Normal Lines

Suppose a surface in $\mathbb{R}^3$ is represented by the equation $F(x,y,z)=0$, and that $P(x_0, y_0, z_0)$ is a point on the surface. We want to write an equation of the plane that is tangent to the surface at point *P*.

Let P be a point on the surface described by $F(x,y,z)=0$.

Suppose $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ describes a curve C that is completely contand in the surface, so that C goes through P.

Then $F(x(t), y(t), z(t)) = 0$ for every t. (because every point on the curve is on the surface)

$\frac{d}{dt}\left(F(x(t), y(t), z(t))\right) = \frac{d}{dt}(0)$

$\frac{dF}{dt} = F'(t) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$ (Chain rule of Section 13.5)

$= \nabla F(x,y,z) \cdot \vec{r}\,'(t) = 0$

∇F is orthogonal to the tangent vector $\vec{r}\,'(t)$ for every curve $\vec{r}(t)$ on the surface

Recall: a plane passing through the point $P(x_0, y_0, z_0)$ can be represented by the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, where $\langle a,b,c \rangle$ is a vector normal to the plane.

So the gradient gives us the normal vector to the surface

Tangent Plane and Normal Line to a Surface:

Suppose a surface *S* is given by $F(x,y,z) = 0$. If *F* is differentiable at the point $P(x_0, y_0, z_0)$ on the surface, and if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then

1. The plane through *P* that is normal to $\nabla F(x_0, y_0, z_0)$ is called the *tangent plane* to *S* at *P*.

2. The line through *P* having the direction of $\nabla F(x_0, y_0, z_0)$ is called the *normal line* to *S* at *P*.

3. The equation of the tangent plane to *S* at *P* is

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

Example 1: Find equations for the tangent plane and the normal line to the surface $x^2 + y^2 + z^2 = 9$ at the point $(1,2,2)$.

Rewrite: $x^2+y^2+z^2 - 9 = 0$ (equation of a sphere)

Let $F(x,y,z) = x^2+y^2+z^2-9$.

$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 2z \rangle$

Normal vector to the tangent plane: $\nabla F(1,2,2) = \langle 2(1), 2(2), 2(2) \rangle = \langle 2,4,4 \rangle$

Equation of plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, where $\vec{N} = \langle a,b,c \rangle$ is a normal vector and (x_0, y_0, z_0) is a point on the plane.

$2(x-1) + 4(y-2) + 4(z-2) = 0$

$2x - 2 + 4y - 8 + 4z - 8 = 0$ eqn of plane

$2x + 4y + 4z - 18 = 0 \Rightarrow \boxed{x + 2y + 2z - 9 = 0}$

Normal line:

$\langle x,y,z \rangle = \langle 1,2,2 \rangle + t\langle 2,4,4 \rangle$

$$\boxed{\begin{aligned} x &= 1+2t, \\ y &= 2+4t, \\ z &= 2+4t \end{aligned}}$$

(eqns of line)

Example 2: Find equations for the tangent plane and the normal line to the surface $z = x^2 - 2xy + y^2$ at the point $(1,2,1)$.

$F(x,y,z) = x^2 - 2xy + y^2 - z = 0$

$\nabla F(x,y,z) = \langle 2x-2y, -2x+2y, -1 \rangle$

$\nabla F(1,2,1) = \langle 2(1)-2(2), -2(1)+2(2), -1 \rangle$

$= \langle -2, 2, -1 \rangle$ (normal vector for tangent plane)

Eqn of plane: $-2(x-1) + 2(y-2) - 1(z-1) = 0$

$-2x + 2 + 2y - 4 - z + 1 = 0$

$\boxed{-2x + 2y - z - 1 = 0}$

(eqn of tangent plane)

Normal line:

$\langle x,y,z \rangle = \langle 1,2,1 \rangle + t\langle -2,2,-1 \rangle$

$$\boxed{\begin{aligned} x &= 1-2t \\ y &= 2+2t \\ z &= 1-t \end{aligned}}$$

symmetric eqns

$$\frac{x-1}{-2} = \frac{y-2}{2} = \frac{z-1}{-1}$$

Example 3: Find an equation for the tangent plane to the surface given by the function $f(x,y) = \ln(x^2+y^2)$, at the point $(1,2)$.

See summer notes

Angle of inclination:

<u>Definition</u>: The angle of inclination of a plane S is the angle θ, with $0 \le \theta \le \frac{\pi}{2}$, between the plane S and the xy-plane.

If a plane has normal vector $\mathbf{n}$, the plane's angle of inclination satisfies the equation

$$\cos\theta = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|\|\mathbf{k}\|} = \frac{|\mathbf{n}\cdot\mathbf{k}|}{\|\mathbf{n}\|}.$$ (because $\|\hat{k}\| = 1$)

<u>Example 4:</u> Find the angle of inclination of the plane that is tangent to the surface given by $2xy - z^3 = 0$ at the point $(2,2,2)$. Let $F(x,y,z) = 2xy - z^3$.

$\nabla F(x,y,z) = \langle 2y, 2x, -3z^2 \rangle$

$\nabla F(2,2,2) = \langle 2(2), 2(2), -3(2)^2 \rangle = \langle 4, 4, -12 \rangle$ (<u>Note</u>: could use $\langle 1,1,-3 \rangle$ as normal vector)

$$\cos\theta = \frac{|\langle 4,4,-12\rangle \cdot \langle 0,0,1\rangle|}{\sqrt{16+16+144}\ (1)} = \frac{|0+0-12|}{\sqrt{176}} = \frac{12}{\sqrt{176}} \quad \frac{12}{\sqrt{16\cdot 11}} = \frac{12}{4\sqrt{11}}$$
$$= \frac{3}{\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{11}}\right) \approx \boxed{25.24^\circ}$$

<u>Example 5:</u> Find the point(s) on the surface $z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$ at which the tangent plane is horizontal.

For the tangent plane to be horizontal, the angle of inclination θ must be 0. So $\cos\theta = 1$.
(Equivalently, the normal vector must be parallel to $\hat{k}$)

$F(x,y,z) = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4 - z = 0$

$\nabla F(x,y,z) = \langle 8x + 4y + 8, 4x - 4y - 5, -1 \rangle$

For $\vec{n} = \nabla F$ to be parallel to $\hat{k}$, we need $F_x = 0$ <u>and</u> $F_y = 0$.

$$\left.\begin{aligned} 8x + 4y + 8 &= 0 \\ 4x - 4y - 5 &= 0 \end{aligned}\right\}$$ System of equations in 2 variables. Solve by elimination or substitution.

Add eqns: $12x + 3 = 0$

$12x = -3$

$x = -\frac{3}{12} = -\frac{1}{4}$

see next page

Ex 5 cont'd:

Put $x = -\frac{1}{4}$ into $8x + 4y + 8 = 0$:

$$8\left(-\frac{1}{4}\right) + 4y + 8 = 0$$
$$-2 + 4y + 8 = 0$$
$$4y + 6 = 0$$
$$4y = -6$$
$$y = -\frac{6}{4} = -\frac{3}{2}$$

For point on the surface, we need to find z:

$$z = 4x^2 + 4xy - 2y^2 + 8x - 5y - 4$$

$x = -\frac{1}{4}$, $y = -\frac{3}{2} \Rightarrow z = 4\left(-\frac{1}{4}\right)^2 + 4\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right) - 2\left(-\frac{3}{2}\right)^2 + 8\left(-\frac{1}{4}\right) - 5\left(-\frac{3}{2}\right) - 4$

$$= 4\left(\frac{1}{16}\right) + \frac{3}{2} - 2\left(\frac{9}{4}\right) - 2 + \frac{15}{2} - 4$$
$$= \frac{1}{4} + \frac{6}{4} - \frac{18}{4} - \frac{8}{4} + \frac{30}{4} - \frac{16}{4}$$
$$= -\frac{5}{4}$$

$\boxed{\text{Tangent plane is horizontal at the point } \left(-\frac{1}{4}, -\frac{3}{2}, -\frac{5}{4}\right).}$