13.8: Extrema of Functions of Two Variables

Definition:

Extreme Values:

The values $f(a,b)$ and $f(c,d)$ are called the *minimum and maximum values*, respectively, of f in a region *R* if $f(a,b) \le f(x,y) \le f(c,d)$ for every (x,y) in *R*.

(For clarity, sometimes these are called *absolute* or *global* extreme values, to distinguish them from relative (local) extreme values.)

Relative Extreme Values:

A function $f(x,y)$ has a *relative (local) maximum* at (x_0,y_0) if $f(x,y) \le f(x_0,y_0)$ for all points (x,y) in an open disk containing (x_0,y_0). The value $f(x_0,y_0)$ is called a *relative maximum* (or *local maximum*) of *f*.

A function $f(x,y)$ has a *relative (local) minimum* at (x_0,y_0) if $f(x,y) \ge f(x_0,y_0)$ for all points (x,y) in an open disk containing (x_0,y_0). The value $f(x_0,y_0)$ is called a *relative minimum* (or *local minimum*) of *f*.

Extreme Value Theorem:

Suppose $f(x,y)$ is a continuous function defined on a closed and bounded region *R* in the *xy*-plane. Then,

1. There is at least one point in *R* at which f takes on a minimum value.
2. There is at least one point in *R* at which f takes on a maximum value.

Definition: Critical Point

Let *f* be defined on an open region containing (x_0,y_0). The point (x_0,y_0) is a *critical point* if one of the following statements is true.

1. $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$ ← (so the tangent plane is horizontal)

OR

2. $f_x(x_0,y_0)$ does not exist, or $f_y(x_0,y_0)$ does not exist.

Example 1: Find the critical points for the function $f(x,y)=(x^2+y^2)^{2/3}$.

Domain is $\mathbb{R}^2$

$f(x,y) = (x^2+y^2)^{2/3}$

$f_x(x,y) = \frac{2}{3}(x^2+y^2)^{-1/3}(2x) = \frac{4x}{3\sqrt[3]{x^2+y^2}}$

$f_y(x,y) = \frac{2}{3}(x^2+y^2)^{-1/3}(2y) = \frac{4y}{3\sqrt[3]{x^2+y^2}}$

Where are f_x, f_y undefined?

Where are f_x and f_y both 0?

f_x and f_y are undefined at the point $(0,0)$. So $(0,0)$ is a critical point.

There are no points where $f_x = f_y = 0$ (because for the numerators of f_x and f_y to be 0, we would need $x=y=0$, which would cause f_x and f_y to be undefined.

Only critical point is $(0,0)$.

Theorem:

If *f* has a relative minimum or relative maximum at the point (x_0, y_0), then (x_0, y_0) must be a critical point of *f*.

This means that if both of the first-order partial derivatives exist at a relative extremum, then the tangent plane must be horizontal.

rel. max or rel. min

Note: As with functions of one variable, not every critical point yields a relative extremum.

Example 2: Find the critical points and relative extrema for the function.

$$f(x,y)=2x^2+y^2+8x-6y+20$$

$f_x(x,y) = 4x+8$

$f_y(x,y) = 2y-6$

set these equal to 0:

$4x+8=0$

$2y-6=0$

$4x=-8$, $x=-2$ | $2y=6$, $y=3$

The point $(-2,3)$ is the only critical point.

(f_x and f_y exist everywhere)

What is behavior of f?

$f(x,y) = 2x^2+8x+y^2-6y+20$

Complete the square:

$f(x,y) = 2(x^2+4x)+(y^2-6y)+20$

$= 2(x^2+4x+4)+(y^2-6y+9)+20-8-9$

$= 2(x+2)^2+(y-3)^2+3$

At $x=-2, y=3$, $f(x,y)=0+0+3=3$.

For every other (x,y), $f(x,y)>3$

Relative minimum is $f(-2,3)=3$.

(Also the absolute minimum of f)

Example 3: Find the critical points and relative extrema for the function.

$g(x, y) = 1 - \sqrt[3]{x^2 + y^2} = 1 - (x^2+y^2)^{1/3}$ Domain: $\mathbb{R}^2$

$g_x(x,y) = -\frac{1}{3}(x^2+y^2)^{-2/3}(2x) = -\frac{2x}{3(x^2+y^2)^{2/3}} = -\frac{2x}{3\sqrt[3]{(x^2+y^2)^2}}$

$g_y(x,y) = -\frac{1}{3}(x^2+y^2)^{-2/3}(2y) = -\frac{2y}{3(x^2+y^2)^{2/3}} = -\frac{2y}{3\sqrt[3]{(x^2+y^2)^2}}$

$\boxed{\text{Critical point: } (0,0)}$ because f_x and f_y are undefined at $(0,0)$.

$g(0,0) = 1 - \sqrt[3]{0^2+0^2} = 1$

for all other $(x,y) \in \mathbb{R}^2$, $g(x,y) < 1$

$\boxed{\text{So, } g \text{ has a relative max } g(0,0) = 1. \text{ This is also the absolute maximum.}}$

Example 4: Find the critical points and relative extrema for the function.

$h(x, y) = 2x^2 - 3y^2$

$h_x(x,y) = 4x$
$h_y(x,y) = -6y$ } defined everywhere

Setting $h_x = 0$ and $h_y = 0$ gives us $x=0, y=0$.

$\boxed{\text{Only critical point is } (0,0).}$

Along the x-axis $(y=0)$, we have $h(x,0) = 2x^2 - 3(0)^2 = 2x^2 \geq 0$ for all x.

Note: $h(0,0) = 0$. For $x \neq 0$, $2x^2 > 0$

Along the y-axis $(x=0)$, we have $h(0,y) = 2(0)^2 - 3y^2 = -3y^2 \leq 0$ for all y.

For $y \neq 0$, $-3y^2 < 0$.

So, h does not have a relative max or min at $(0,0)$. $\boxed{\text{No relative extrema}}$

Example 5: Find the critical points and relative extrema for the function.

$$h(x,y)=2x^2-3y^2$$

Same as Ex 4.

Theorem: Second Partials Test

Suppose the second partial derivatives of $f(x,y)$ are continuous on an open region containing the point (a,b).

Suppose also that $f_x(a,b)=0$ and $f_y(a,b)=0$. so (a,b) is a critical point

Define $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$. Then, $D = f_{xx}f_{yy} - f_{xy}^2$

1. If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then f has a relative minimum at (a,b).
2. If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then f has a relative maximum at (a,b).
3. If $D(a,b)<0$, then there is a saddle point (and thus, no relative extremum) at (a,b).
4. If $D(a,b)=0$, then the second partials test is inconclusive.

Recall: $f_{xy}=f_{yx}$ if all partials are continuous

Note: D can be written as a determinant:

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy}-f_{xy}f_{yx}=f_{xx}f_{yy}-\left(f_{xy}\right)^2$$

Example 6: Find and classify the critical points of $f(x,y) = -x^3 + 4xy - 2y^2 + 1$.

$f_x(x,y) = -3x^2 + 4y$

$f_y(x,y) = 4x - 4y$

Set $f_x = 0, f_y = 0$: $-3x^2 + 4y = 0$

$4x - 4y = 0$

$4x = 4y$

$x = y$

Put $x = y$ into $-3x^2 + 4y = 0$:

$-3y^2 + 4y = 0$

$y(-3y+4) = 0 \Rightarrow y = 0,\ y = \frac{4}{3}$

$y = 0 \Rightarrow x = 0$ because $y = x$

$y = \frac{4}{3} \Rightarrow x = \frac{4}{3}$ because $y = x$

2 critical points: $(0,0)$, $(\frac{4}{3}, \frac{4}{3})$

$f_{xx} = -6x$

$f_{yy} = -4$

$f_{xy} = 4$

$f_{yx} = 4$ $\checkmark$

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Example 7: Find and classify the critical points of $f(x,y) = 3x^2 + 12xy + 2y^2 + 6y + 5$.

$f_x(x,y) = 6x + 12y$

$f_y(x,y) = 12x + 4y + 6$

Set $f_x = f_y = 0$: $6x + 12y = 0 \xrightarrow{(-2)} -12x - 24y = 0$

$12x + 4y + 6 = 0 \longrightarrow 12x + 4y + 6 = 0$

Add $-20y + 6 = 0$

$6 = 20y$

$\frac{6}{20} = y$

$y = \frac{3}{10}$

$6x + 12y = 0,\ y = \frac{3}{10} \Rightarrow 6x + 12\left(\frac{3}{10}\right) = 0$

$6x = -\frac{36}{10}$

$6x = -\frac{18}{5}$

$x = -\frac{18}{30} = -\frac{3}{5}$

Critical Point: $\left(-\frac{3}{5}, \frac{3}{10}\right)$

$f_{xx}(x,y) = 6$

$f_{yy}(x,y) = 4$

$f_{xy}(x,y) = 12$

$f_{yx}(x,y) = 12$ $\overset{\checkmark}{=}$

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2$

$= 6(4) - 12^2 = -120 < 0$

$\boxed{\text{Saddle Point (no relative max/min) at } \left(-\frac{3}{5}, \frac{3}{10}\right)}$

Ex 6 Continued:

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$D(0,0) = -6(0)(-4) - 4^2 = 0 - 4^2 = -16 < 0$$

$$\boxed{\text{Saddle point at } (0,0)}$$

$$D\left(\tfrac{4}{3}, \tfrac{4}{3}\right) = -6\left(\tfrac{4}{3}\right)(-4) - 4^2$$
$$= -8(-4) - 16 = 32 - 16 = 16 > 0$$

$$f_{xx}\left(\tfrac{4}{3}, \tfrac{4}{3}\right) = -6\left(\tfrac{4}{3}\right) = -8 < 0$$

$$\boxed{\text{So relative max at } \left(\tfrac{4}{3}, \tfrac{4}{3}\right).}$$

Example 8: Find and classify the critical points of $f(x,y) = x^3 + 2xy^2 + 5x^2 + 4y^2 + 3$.

See Summer notes

Example 9: Find and classify the critical points of $f(x,y) = 2xy - \frac{1}{2}x^4 - -\frac{1}{2}y^4 + 1$.

$f(x,y) = 2xy - \frac{1}{2}x^4 - \frac{1}{2}y^4 + 1$

$f_x(x,y) = 2y - 2x^3$

$f_y(x,y) = 2x - 2y^3$

$f_{xx}(x,y) = -6x^2$

$f_{yy}(x,y) = -6y^2$

$f_{xy}(x,y) = 2$

$f_{yx}(x,y) = 2$ ✓

$D = f_{xx}f_{yy} - f_{xy}^2$

$D(x,y) = -6x^2(-6y^2) - 2^2$

$= 36x^2y^2 - 4$

$D(0,0) = -4 < 0 \Rightarrow$ saddle pt

$D(1,1) = 36(1)^2(1)^2 - 4 > 0$
$f_{xx}(1,1) = -6(1)^2 < 0$ } relative max

$D(-1,-1) = 36 - 4 > 0$, $f_{xx}(-1,-1) < 0$ relative max

Set $f_x = f_y = 0$:

$2y - 2x^3 = 0 \Rightarrow 2y = 2x^3 \Rightarrow y = x^3$

$2x - 2y^3 = 0$

$y = x^3 \Rightarrow 2x - 2(x^3)^3 = 0$

$2x - 2x^9 = 0$

$2x(1 - x^8) = 0$

$2x = 0 \quad | \quad 1 - x^8 = 0$

$x = 0 \quad | \quad 1 = x^8$

$\qquad\quad | \quad x = \pm 1$

Back to $y = x^3$:

$x = 0 \Rightarrow y = 0^3 = 0$

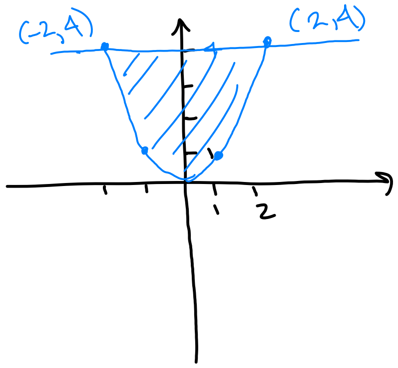
$x = 1 \Rightarrow y = 1^3 = 1$

$x = -1 \Rightarrow y = (-1)^3 = -1$

Three critical points: $(0,0)$, $(1,1)$, $(-1,-1)$

Saddle Point: $(0,0)$
Relative max at $(1,1)$ and at $(-1,-1)$

Example 10: Find the absolute extrema of $f(x,y)=3x^2+2y^2-4y$ over the region bounded by the graphs of $y=x^2$ and $y=4$.



Find intersection points: set y's equal: $x^2=4$

$x=\pm 2$

$f_x(x,y)=6x$

$f_y(x,y)=4y-4$

$f_{xx}(x,y)=6$

$f_{yy}(x,y)=4$

$f_{xy}(x,y)=0$ equal.

$f_{yx}(x,y)=0$

Find critical points:

Set $f_x=f_y=0$:

$6x=0 \Rightarrow x=0$

$4y-4=0 \Rightarrow 4y=4$

$\Rightarrow y=1$

Critical point: $(0,1)$

Is it in my region? Yes

At $(0,1)$: $D(0,1)=f_{xx}(0,1)f_{yy}(0,1)-[f_{yx}(0,1)]^2$

$=6(4)-0^2=24>0$

$f_{xx}(0,1)>0$, so f has a relative minimum at $(0,1)$

Analyze the boundaries: $f(x,y)=3x^2+2y^2-4y$

Along the line $y=4$: $f(x,4)=3x^2+2(4)^2-4(4)$

$=3x^2+32-16$

$=3x^2+16$

What are the minimum and maximum of f along this boundary?

This gives us the ordered pair $(0,4)$

Minimum value of $f(x,4)=3x^2+16$ is 16, occurring when $x=0$. Is this on the boundary of my region? Yes

The maximum along the top boundary occurs when $x=\pm 2$.

At $x=\pm 2$, $f(x,4)=3(\pm 2)^2+16=3(4)+16=28$

Here, the ordered pairs are $(2,4)$ and $(-2,4)$

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Thus $f(2,4)=f(-2,4)=28$. Max along top boundary

Along the parabolic boundary $y = x^2$: $\quad f(x,y) = 3x^2 + 2y^2 - 4y$

this function becomes
$$f(x,x^2) = 3x^2 + 2(x^2)^2 - 4(x^2)$$
$$= 3x^2 + 2x^4 - 4x^2$$
$$= 2x^4 - x^2$$

We now need to find the minimum and maximum of $g(x) = 2x^4 - x^2$ on the interval $[-2,2]$.

Find critical numbers: $g'(x) = 8x^3 - 2x$
$$= 2x(4x^2 - 1)$$
$$= 2x(2x+1)(2x-1)$$
$x = 0$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$ (critical numbers of $g(x) = 2x^4 - x^2$

$g(0) = 2(0)^4 - 0^2 = 0$

min along this boundary
$$\begin{cases} g(\frac{1}{2}) = 2(\frac{1}{2})^4 - (\frac{1}{2})^2 = 2(\frac{1}{16}) - \frac{1}{4} = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8} \\ g(-\frac{1}{2}) = 2(-\frac{1}{2})^4 - (-\frac{1}{2})^2 = -\frac{1}{8} \text{ also} \end{cases}$$

max along this boundary
$$\begin{cases} g(2) = 2(2)^4 - 2^2 = 32 - 4 = 28 \\ g(-2) = 2(-2)^4 - (-2)^2 = 32 - 4 = 28 \end{cases}$$

Along parabolic boundary, the min is $g(\frac{1}{2}) = g(-\frac{1}{2}) = -\frac{1}{8}$
the max is $g(2) = g(-2) = 28$

Along top boundary, the min was $f(0,4) = 16$
the max was $f(2,4) = f(-2,4) = 28$

In interior, relative min was $f(0,1) = 3(0)^2 + 2(1)^2 - 4(1) = -2$

$$\boxed{\begin{array}{l}\text{The absolute maximum is } f(2,4) = f(-2,4) = 28. \\ \text{The absolute minimum is } f(0,1) = -2.\end{array}}$$