13.9: Applications of Extrema

Example 1: Find three positive numbers x, y, and z such that the sum is 32 and $P = xy^2z$ is at a maximum.

Maximize: $P(x,y,z) = xy^2z$

We want to write P as a function of two variables only.

$x + y + z = 32$

Solve for z: $z = 32 - x - y$

Put this into P: $\hat{P}(x,y) = xy^2(32-x-y)$

$= 32xy^2 - x^2y^2 - xy^3$

$\hat{P}_x(x,y) = 32y^2 - 2xy^2 - y^3$

$\hat{P}_y(x,y) = 64xy - 2x^2y - 3xy^2$

$\hat{P}_{xx}(x,y) = -2y^2$

$\hat{P}_{yy}(x,y) = 64x - 2x^2 - 6xy$

$\hat{P}_{xy}(x,y) = 64y - 4xy - 3y^2$

$\hat{P}_{yx}(x,y) = 64y - 4xy - 3y^2$

equal ✓

Find critical numbers:

Set $\hat{P}_x = \hat{P}_y = 0$:

$32y^2 - 2xy^2 - y^3 = 0$

$64xy - 2x^2y - 3xy^2 = 0$

1st eqn:

$32y^2 - 2xy^2 - y^3 = 0$

$y^2(32 - 2x - y) = 0$

$y^2 = 0$ | $32 - 2x - y = 0$

Throw out $y=0$ (problem specified positive #s) | $32 - 2x = y$

Put this into 2nd eqn

2nd eqn: $64xy - 2x^2y - 3xy^2 = 0$

$xy(64 - 2x - 3y) = 0$

$xy = 0$ | $64 - 2x - 3y = 0$

$x = 0, y = 0$ Throw out | Put in $y = 32 - 2x$:

$64 - 2x - 3(32 - 2x) = 0$

$64 - 2x - 96 + 6x = 0$

$-32 + 4x = 0$

$4x = 32$

$x = 8$

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Then $y = 32 - 2x = 32 - 2(8) = 16$. So critical pt is $(8, 16)$

Ex 1 cont'd

2nd partials test:

$$D = \hat{P}_{xx}\hat{P}_{yy} - \hat{P}_{xy}^2$$

$\hat{P}_{xx}(x,y) = -2y^2 \Rightarrow \hat{P}_{xx}(8,16) = -2(16)^2 = -2(256) = -512$

$\hat{P}_{yy}(x,y) = 64x - 2x^2 - 6xy \Rightarrow \hat{P}_{yy}(8,16) = 64(8) - 2(8)^2 - 6(8)(16)$
$= -384$

$\hat{P}_{xy}(x,y) = 64y - 4xy - 3y^2 \Rightarrow \hat{P}_{xy}(8,16) = 64(16) - 4(8)(16) - 3(16)^2$
$= -256$

$D(8,16) = -512(-384) - (-256)^2 > 0$

$P_{xx}(8,16) < 0$, so there is a max at $(8,16)$

we have $x = 8$, $y = 16$.

From earlier, $z = 32 - x - y = 32 - 8 - 16 = 8$

$\boxed{\text{So the three positive numbers are } x=8,\ y=16,\ z=8}$

The maximum P is $P(8,16,8) = 8(16)^2(8) = 16384$.

or, you could write: $\boxed{\text{The maximum } P \text{ occurs at } (x,y,z) = (8,16,8).}$

Example 2: Melika Candle Company manufactures candles at two locations. At Location A, the cost of producing x units is $C_A = 0.02x^2 + 4x + 500$. At Location B, the cost of producing x units is $C_B = 0.05x^2 + 4x + 275$. The candles sell for \$15 per unit. Find the quantity that should be produced at each location to maximize profit.

Maximize: Profit = P

Let x = number of candles made at Location A,
y = number of candles made at Location B.

Profit = Revenue − Cost

Revenue at Location A: $15x$
Cost at Location A: $0.02x^2 + 4x + 500$

Revenue at Location B: $15y$
Cost at Location B: $0.05y^2 + 4y + 275$

Total Profit = Rev A + Rev B − Cost A − Cost B

$$P(x, y) = 15x + 15y - 0.02x^2 - 4x - 500 - 0.05y^2 - 4y - 275$$

See summer notes

Example 3: Find the distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$.

Minimize: Distance from $(1,0,-2)$ to a point $P(x,y,z)$ on the plane. Distance between $(1,0,-2)$ and (x,y,z) is

$$d=\sqrt{(x-1)^2+(y-0)^2+(z-(-2))^2}$$. We want to minimize d.

$$d=\sqrt{(x-1)^2+y^2+(z+2)^2}$$

Write it as a function of x and y only: Solve plane eqn for z:

$$x+2y+z=4 \implies z=4-x-2y$$

Put $z=4-x-2y$ into eqn for d:

$$d=\sqrt{(x-1)^2+y^2+(4-x-2y+2)^2}$$

Instead of minimizing d, we can minimize d^2:

Let $m=d^2=(x-1)^2+y^2+(6-x-2y)^2$ Minimize m

$$m_x(x,y)=2(x-1)(1)+2(6-x-2y)(-1)$$
$$=2x-2-12+2x+4y$$
$$=4x+4y-14$$

$$m_y(x,y)=2y+2(6-x-2y)(-2)$$
$$=2y-24+4x+8y$$
$$=4x+10y-24$$

$m_{xx}(x,y)=4$

$m_{yy}(x,y)=10$

$m_{xy}(x,y)=4$ $\stackrel{=}{\checkmark}$

$m_{yx}(x,y)=4$

Find critical points: Set $m_x=m_y=0$:

$$4x+4y-14=0$$
$$4x+10y-24=0$$

Subtract: $-6y+10=0$

$$10=6y$$
$$\frac{5}{3}=y$$

Put $y=\frac{5}{3}$ into $4x+4y-14=0$:

$$4x+4\left(\frac{5}{3}\right)-14=0$$
$$4x+\frac{20}{3}-\frac{42}{3}=0$$

$$4x=\frac{22}{3}$$
$$x=\frac{22}{3}\left(\frac{1}{4}\right)$$
$$x=\frac{22}{12}$$
$$=\frac{11}{6}$$

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Ex 3 cont'd: So only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$

$$D = M_{xx} M_{yy} - (M_{xy})^2$$

$$D\left(\tfrac{11}{6}, \tfrac{5}{3}\right) = 4(10) - 4^2 = 24 > 0$$

Check sign of M_{xx}:

$M_{xx}\left(\tfrac{11}{6}, \tfrac{5}{3}\right) = 4 > 0$, so there is a minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$

Find z: $z = 4 - x - 2y$

$$z = 4 - \tfrac{11}{6} - 2\left(\tfrac{5}{3}\right)$$

$$= \tfrac{24}{6} - \tfrac{11}{6} - \tfrac{20}{6} = -\tfrac{7}{6}$$

Closest point on the plane to $(1, 0, -2)$ is $\left(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6}\right)$.

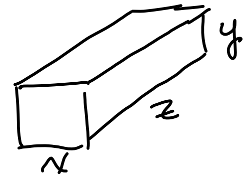
The minimum distance is given by $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$

$$= \sqrt{\left(\tfrac{11}{6} - \tfrac{6}{6}\right)^2 + \left(\tfrac{5}{3}\right)^2 + \left(-\tfrac{7}{6} + \tfrac{12}{6}\right)^2}$$

$$= \sqrt{\tfrac{25}{36} + \tfrac{100}{36} + \tfrac{25}{36}}$$

$$= \sqrt{\tfrac{150}{36}} = \frac{\sqrt{150}}{6} = \frac{\sqrt{25 \cdot 6}}{6} = \boxed{\frac{5\sqrt{6}}{6}}$$

Example 4: A rectangular box without a lid is to be made from 12 square meters of cardboard. Find the maximum volume of such a box.



Maximize: $V = xyz$

We want to write V in terms of 2 variables only.

Surface Area $= 12 = 2xy + 2yz + xz$

solve for z:
$$12 - 2xy = 2yz + xz$$
$$12 - 2xy = z(2y + x)$$
$$\frac{12-2xy}{2y+x} = z$$

Solve for y:
$$12 - xz = 2xy + 2yz$$
$$12 - xz = y(2x + 2z)$$
$$\frac{12-xz}{2x+2z} = y$$

← I will choose to work with this one, because it has symmetry between variables (exchanging x and z does not cause a change in y)

$$V = xyz = x\left(\frac{12-xz}{2x+2z}\right)(z)$$

$$V(x,z) = \frac{12xz - x^2z^2}{2x+2z}$$

$$V_x(x,z) = \frac{(2x+2z)(12z - 2xz^2) - (12xz - x^2z^2)(2)}{(2x+2z)^2}$$

$$= \frac{\cancel{24xz} - 4x^2z^2 + 24z^2 - 4xz^3 - \cancel{24xz} + 2x^2z^2}{(2x+2z)^2}$$

$$= \frac{-2x^2z^2 + 24z^2 - 4xz^3}{(2(x+z))^2} = \frac{2z^2(-x^2 + 12 - 2xz)}{4(x+z)^2}$$

$$= \frac{z^2(-x^2 + 12 - 2xz)}{2(x+z)^2}$$

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Because of the symmetry in the roles of x and z in V, I can exchange positions of variables to write V_z:

$$V_x(x,z) = \frac{z^2(-x^2+12-2xz)}{2(x+z)^2}$$

$$V_z(x,z) = \frac{x^2(-z^2+12-2xz)}{2(x+z)^2}$$

Find critical Pts.: Set $V_x = V_z = 0$. So numerators must be 0.

Domain of V_x, V_z: not defined for $x=0, z=0$, but these will give me a zero-volume box.

Numerators: $z^2(-x^2+12-2xz)=0$
$x^2(-z^2+12-2xz)=0$

If $z^2=0$ or $x^2=0$, we don't have a box.

So, $-x^2+12-2xz=0$
$-z^2+12-2xz=0$

Subtract eqns: $-x^2+z^2=0$

$$z^2 = x^2$$

$$x = \pm z$$

a negative value for x or z does not give us a box.

So, $x=z$

$$-x^2+12-2xz=0$$

$x=z \Longrightarrow -x^2+12-2x(x)=0$

$$-x^2+12-2x^2=0$$

$$12-3x^2=0$$

$$12=3x^2$$

$$4=x^2$$

$x=\pm 2$ throw out -2, so $x=2$

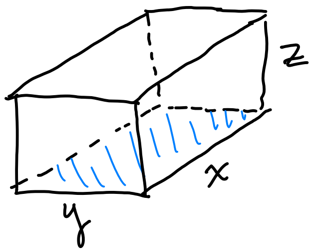
Then $z=2$

Find y: From earlier, $y = \frac{12-xz}{2x+2z} = \frac{12-2(2)}{2(2)+2(2)} = \frac{8}{8} = 1$

So the maximum Volume is $2m(2m)(1m) = \boxed{4m^3}$
(occurs when height is 1m and length and depth are 2m)

Example 5: The base of an aquarium with given volume V is made of slate and the sides are made of glass. If slate costs five times as much (per unit area) as glass, find the dimensions of the aquarium that minimize the cost of the materials. (It didn't mention a top, so I will assume the top is open).

(This was not worked during class).



Cost of sides: k/unit^2

cost of base: $5k/\text{unit}^2$

Total Cost: $C = 2kxz + 2kyz + 5kxy$

We want to write C as a function of x and y only. Put in $z = \frac{V}{xy}$.

$V = xyz$

$z = \frac{V}{xy} \implies$

$$C(x,y) = 2kx\left(\frac{V}{xy}\right) + 2ky\left(\frac{V}{xy}\right) + 5kxy$$

$$= \frac{2kxV + 2kyV + 5kx^2y^2}{xy}$$

$$C_x(x,y) = \frac{(xy)(2kV + 10kxy^2) - (2kxV + 2kyV + 5kx^2y^2)(y)}{(xy)^2}$$

$$= \frac{2k\cancel{V}xy + 10kx^2y^3 - 2k\cancel{xyV} - 2ky^2V - 5kx^2y^3}{x^2y^2}$$

$$= \frac{5kx^2y^3 - 2ky^2V}{x^2y^2} = \frac{5kx^2y - 2kV}{x} \quad \text{(dividing by } y^2\text{)}$$

Symmetry of variables $\implies C_y(x,y) = \frac{5ky^2x - 2kV}{y}$

Note: we know $x \neq 0$, $y \neq 0$ (because they are dimensions of a fish tank)

$C_x = 0 \implies 5kx^2y - 2kV = 0$
$5kx^2y = 2kV$

$C_y = 0 \implies 5ky^2x - 2kV = 0$
$5ky^2x = 2kV$

Setting $2kV$'s equal: $5kx^2y = 5ky^2x$

$x^2y = y^2x$

$x^2y - y^2x = 0$

$xy(x-y) = 0$

$\underbrace{x = 0,\ y = 0}_{\text{throw out}},\ x = y$

See next page

Ex 5 cont'd: Put $x=y$ into $5kx^2y-2kV=0$

$$5kx^2(x)-2kV=0$$

$$5kx^3=2kV$$

$$x^3=\frac{2kV}{5k}=\frac{2V}{5}$$

$$x=\sqrt[3]{\frac{2V}{5}}$$

$x=y \Rightarrow y=\sqrt[3]{\frac{2V}{5}}$ also

Then $z=\frac{V}{xy}=\frac{V}{\sqrt[3]{\frac{2V}{5}}\sqrt[3]{\frac{2V}{5}}}=\frac{V}{\sqrt[3]{\frac{4V^2}{25}}}=\frac{V\sqrt[3]{25}}{\sqrt[3]{4V^2}}$

$$=\frac{V\sqrt[3]{25}}{V^{2/3}\sqrt[3]{4}}=\frac{V^{1/3}\sqrt[3]{25}}{\sqrt[3]{4}}=\sqrt[3]{\frac{25V}{4}}$$

$$\boxed{\text{To minimize the cost of materials, the base of the aquarium should measure } \sqrt[3]{\tfrac{2V}{5}} \times \sqrt[3]{\tfrac{2V}{5}} \text{ unit}^2\text{, and the height should be } \sqrt[3]{\tfrac{25V}{4}}.}$$