14.1: Iterated Integrals and Area in the Plane

Suppose *f* is a function of two variables that is continuous on the rectangle $R = [a,b] \times [c,d]$.

The notation $\int_c^d f(x,y)\, dy$ means that we consider *x* to be fixed (constant), and we integrate $f(x,y)$ with respect to y from $y = c$ to $y = d$. This is called *partial integration*. The result is a function of *x*:

$$A(x) = \int_c^d f(x,y)\, dy.$$

This new function $A(x)$ can be integrated with respect to *x* from $x = a$ to $x = b$, resulting in:

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

We omit the brackets and write $\int_a^b \int_c^d f(x,y)\, dy\, dx$, called an *iterated integral*.

Note: The order of integration is "from the inside out."

Similarly, the notation $\int_a^b f(x,y)\, dx$ means that we consider *y* to be fixed (constant), and we integrate $f(x,y)$ with respect to *x* from $x = a$ to $x = b$. The result is a function of *y*:

$$B(y) = \int_a^b f(x,y)\, dx.$$

This new function $B(y)$ can be integrated with respect to *y* from $y = c$ to $y = d$, resulting in:

$$\int_c^d B(y)\, dx = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

Example 1: Calculate $\int_x^{x^2} \frac{y}{x}\, dy$.

Example 2: Calculate $\int_y^{y^2} \frac{y}{x}\, dx$.

Example 3: Calculate $\int_0^2 \int_1^3 2x^2 y^3 \, dy \, dx$ and $\int_1^3 \int_0^2 2x^2 y^3 \, dx \, dy$.

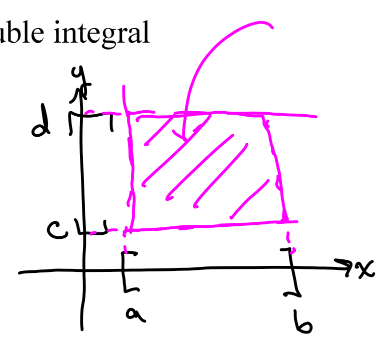
> <u>Definition</u>: (Double Integral) (See Section 14.2 in Larson book.)
>
> Suppose $f(x, y)$ is defined on a closed, bounded region *R* in the plane. Also suppose that *R* is partitioned into *n* rectangles in such a way that the norm of the partition (diagonal of the largest rectangle, denoted $\|\Delta\|$) approaches 0 as the number of rectangles approaches infinity. (In other words, $\|\Delta\| \to 0$ as $n \to \infty$).
> Then the double integral of *f* over *R* is
>
> $$\iint_R f(x, y) \, dA = \lim_{\|\Delta\| \to 0, n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \, \Delta A_i .$$
>
> where ΔA_i is the area of the *i*th rectangle, and (x_i, y_i) is any point in the *i*th rectangle (provided this limit exists).

Using double integrals to find area:

To find area of a region, we integrate the constant function $f(x, y) = 1$. (Because if $f(x, y) = 1$, then $f(x_i, y_i) \, \Delta A_i = \Delta A_i$. If we add up all the areas ΔA_i, we can approximate the area of our region.)

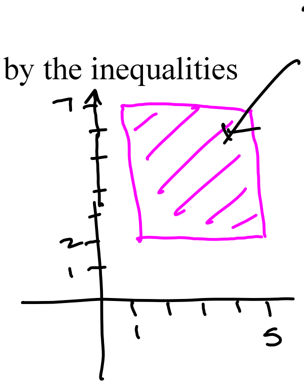
$[a,b] \times [c,d]$

We'll also need the following theorem, which allows us to break down our double integral $\iint_R f(x, y) \, dA$ into an iterated integral using dx and dy.



> <u>Fubini's Theorem</u>: (Minor League Version)
>
> If $f(x, y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$, then
>
> $$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy .$$

Example 4: Use an iterated integral to find the area of the region described by the inequalities $1 \le x \le 5$, $2 \le y \le 7$. To find area, integrate $f(x,y) = 1$.



$$\text{Area} = \iint_R 1\, dA = \int_2^7 \int_1^5 1\, dx\, dy$$

$$= \int_2^7 x \Big|_1^5 dy = \int_2^7 (5-1)\, dy = \int_2^7 4\, dy$$

$$= 4y \Big|_2^7 = 4(7) - 4(2) = \boxed{20}$$

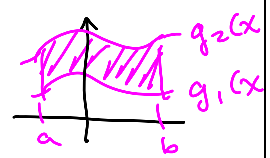
Geometry: width $= 5 - 1 = 4$
height $= 7 - 2 = 5$
Area $= 4(5) = 20$ ✓

Theorem: (Area of a Region)

The area of the region bounded by the graphs of $y = g_1(x)$, $y = g_2(x)$, $x = a$, $x = b$ is given by

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx, \qquad \text{provided } g_1 \text{ and } g_2 \text{ are continuous..}$$

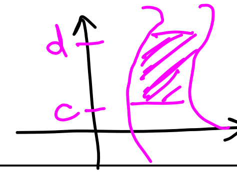
$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$



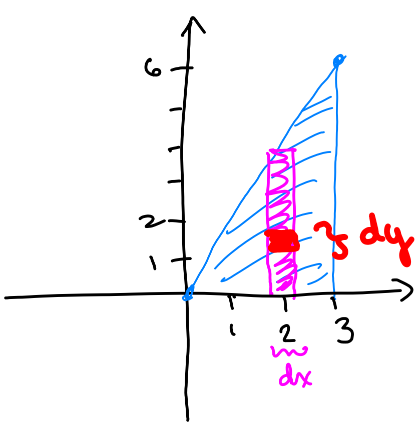
The area of the region bounded by the graphs of $x = h_1(y)$, $x = h_2(y)$, $y = c$, $y = d$ is given by

$$\int_c^d \int_{h_1(y)}^{h_1(y)} dx\, dy, \qquad \text{provided } h_1 \text{ and } h_2 \text{ are continuous.}$$

$\int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy$



Example 5: Find the area of the triangle bounded by the graphs of $y = 2x$, $y = 0$, and $x = 3$.



$$\text{Area} = \int_0^3 \int_0^{2x} 1\, dy\, dx$$

$$= \int_0^3 y \Big|_0^{2x} dx = \int_0^3 (2x - 0)\, dx$$

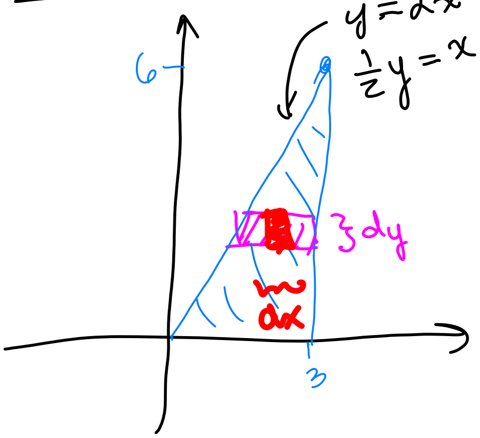
$$= \int_0^3 2x\, dx = \frac{2x^2}{2}\Big|_0^3 = x^2 \Big|_0^3$$

$$= 3^2 - 0 = \boxed{9}$$

Geometry:
Area $= \frac{1}{2}(\text{base})(\text{height})$
$= \frac{1}{2}(3)(6) = \boxed{9}$ ✓

Try setting it up in the opposite order: See next page.

Ex 5 (changing order)



$$\text{Area} = \int_0^6 \int_{\frac{1}{2}y}^{3} 1 \, dx \, dy$$

$$= \int_0^6 x \Big|_{\frac{1}{2}y}^{3} \, dy$$

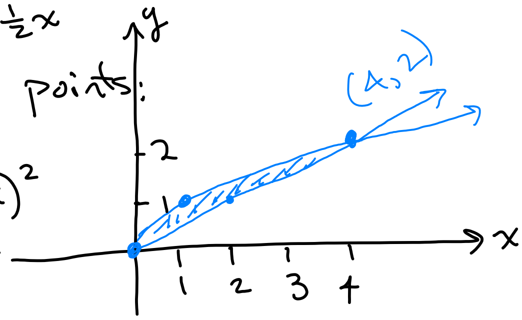
$$= \int_0^6 \left(3 - \tfrac{1}{2}y\right) dy$$

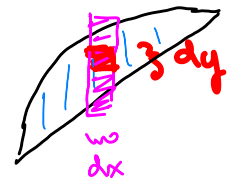
$$= \left[3y - \frac{1}{2}\frac{y^2}{2}\right]_0^6 = 3(6) - \frac{6^2}{4} = 18 - 9 = \boxed{9}$$

Example 6: Set up integrals to find the area of the region bounded by the graphs of $y=\sqrt{x}$ and ~~$x = y^2$~~. $y = \frac{1}{2}x$

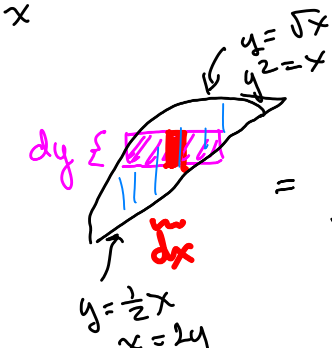
Find the intersection points:

$$\sqrt{x} = \frac{1}{2}x$$
$$(\sqrt{x})^2 = \left(\frac{1}{2}x\right)^2$$
$$x = \frac{1}{4}x^2$$
$$4x = x^2$$
$$0 = x^2 - 4x$$
$$0 = x(x-4)$$
$$x = 0, \quad x = 4$$
$$x = 4 \Rightarrow y = \frac{1}{2}x = \frac{1}{2}(4) = 2$$





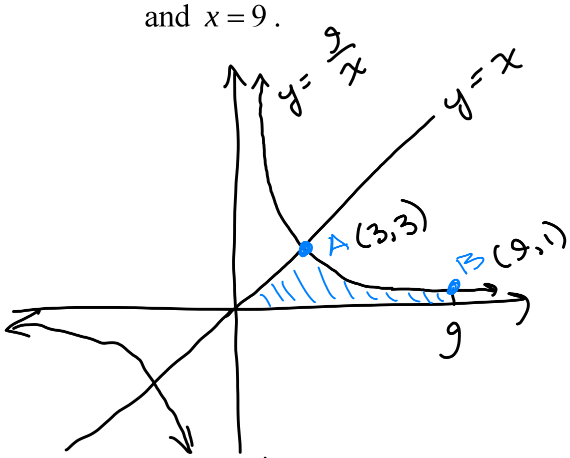
$$\text{Area} = \int_0^4 \int_{\frac{1}{2}x}^{\sqrt{x}} 1 \, dy \, dx$$



$$\text{Area} = \int_0^2 \int_{y^2}^{2y} 1 \, dx \, dy$$

Example 7: Find the area of the region bounded by the graphs of $y = 2x$ and $y = x^{3/2}$.

Example 8: Find the area of the region bounded by the graphs of $xy = 9$, $y = x$ and $y = 0$, and $x = 9$.



Find intersection pt A:

$$\frac{9}{x} = x$$
$$9 = x^2$$
$$x = \pm 3$$

we only care about $x = +3$.

$$y = \frac{9}{x} = \frac{9}{3} = 3$$

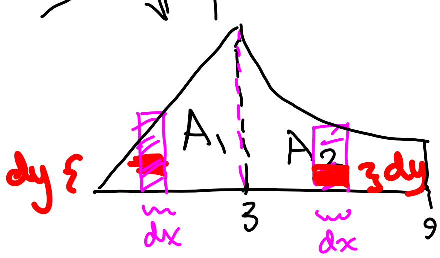
Pt A is $(3, 3)$.

$y = \frac{9}{x}$

Find intersection pt B:

$$x = 9, \quad y = \frac{9}{x}$$
$$x = 9 \Rightarrow y = \frac{9}{x} = \frac{9}{9} = 1$$

Pt B is $(9, 1)$



$$\text{Area} = A_1 + A_2 = \int_0^3 \int_0^x 1 \, dy \, dx + \int_3^9 \int_0^{9/x} 1 \, dy \, dx$$

Switching the order of integration:

Example 9: For the given iterated integral, sketch the region of integration and then switch the order of integration.

Current integral uses horizontal rectangles ✓

$$I = \int_0^4 \int_{\sqrt{y}}^2 f(x,y)\, dx\, dy$$

Boundaries of region:

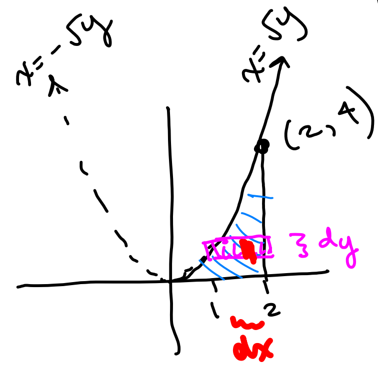
$x = \sqrt{y}$

$x = 2$

dx runs from $x = \sqrt{y}$ to $x = 2$

$x = \sqrt{y}$ is one half of $x^2 = y$

original order of integration →



$$\boxed{I = \int_0^2 \int_0^{x^2} f(x,y)\, dy\, dx}$$

Example 10: For the given iterated integral, sketch the region of integration and then switch the order of integration.

$$\int_0^4 \int_0^{4-x^2} f(x,y)\, dy\, dx$$

Example 11: Sketch the region R whose area is given by the iterated integral. Switch the order of integration and show that both orders yield the same area.

$$\int_{-2}^2 \int_0^{4-y^2} dx\, dy$$

Example 12: Sketch the region R whose area is given by the iterated integral. Switch the order of integration and show that both orders yield the same area.

$$\underbrace{\int_0^4 \int_0^{x/2} dy\, dx}_{I_1} + \underbrace{\int_4^6 \int_0^{6-x} dy\, dx}_{I_2}$$

Current order:

Find intersection pt:

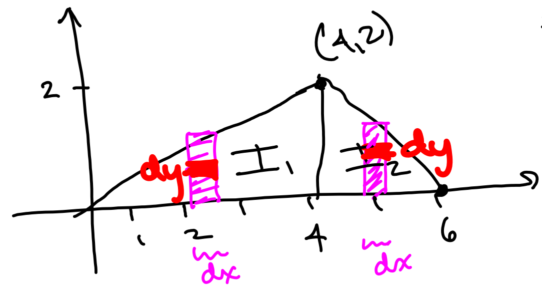
$\frac{x}{2} = 6 - x$

$x = 12 - 2x$

$3x = 12$

$x = 4$

$y = 6 - 4 = 2$



I_1: dy goes from $y = 0$ to $y = \frac{x}{2}$

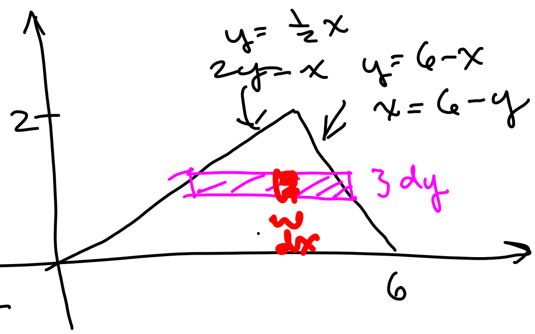
I_2: dy goes from $y = 0$ to $y = 6 - x$

$y = -x + 6$

$y = 0 \Rightarrow 0 = -x + 6$

$x = 6$

$(6, 0)$



$$\text{Area} = \int_0^2 \int_{2y}^{6-y} dx\, dy = \int_0^2 x \Big|_{2y}^{6-y} dy$$

$$= \int_0^2 (6 - y - 2y)\, dy = \int_0^2 (6 - 3y)\, dy$$

$$= \left(6y - \frac{3y^2}{2}\right)\Big|_0^2$$

$$= 6(2) - \frac{3(2)^2}{2} - 0 = 12 - 6 = \boxed{6}$$

Example 13: Calculate the iterated integral.

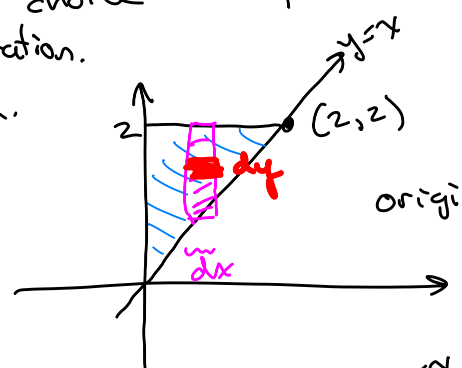
$$\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$$

$$\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$$

e^{-y^2} has no antiderivative created out of elementary functions (to get an antiderivative, you need a power series)

So, we have no choice except to change the order of integration.

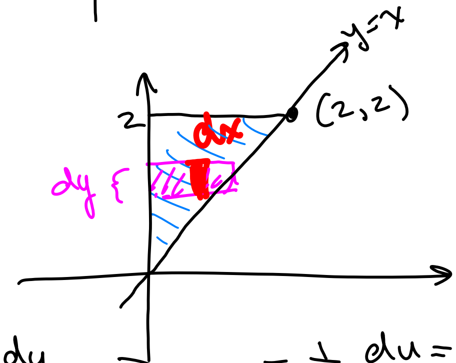
Inner integral: dy runs from $y = x$ to $y = 2$.



original setup

$$\int_0^2 \int_x^2 e^{-y^2}\, dy\, dx$$

$$= \int_0^2 \int_0^y e^{-y^2}\, dx\, dy$$



$$= \int_0^2 x e^{-y^2} \Big|_0^y dy = \int_0^2 \left(y e^{-y^2} - 0 e^{-y^2}\right) dy$$

$$= \int_0^2 y e^{-y^2}\, dy$$

See next page

$u = -y^2$

$\frac{du}{dy} = -2y$

$du = -2y\, dy \Rightarrow -\frac{1}{2} du = y\, dy$

$= \int_0^2 y\,\underbrace{e^{-y^2}}_{e^u}\,\underbrace{dy}_{-\frac{1}{2}du} = -\frac{1}{2}\int_{y=0}^{y=2} e^u\,du$

$= -\frac{1}{2}e^u \Big|_{y=0}^{y=2} = -\frac{1}{2}e^{-y^2}\Big|_0^2$

$= -\frac{1}{2}e^{-4} + \frac{1}{2}e^0$

$= \boxed{-\frac{1}{2}e^{-4} + \frac{1}{2}}$

$= -\frac{1}{2e^4} + \frac{1}{2}\cdot\frac{e^4}{e^4}$

$= \boxed{\frac{e^4-1}{2e^4}}$

Example 14: Calculate the iterated integral.

$$\int_1^2 \int_0^\pi x\cos(xy)\,dx\,dy$$