14.2: Double Integrals and Volume

Definition: (Double Integral)

Suppose $f(x,y)$ is defined on a closed, bounded region *R* in the plane. Also suppose that *R* is partitioned into *n* rectangles in such a way that the norm of the partition (diagonal of the largest rectangle, denoted $\|\Delta\|$) approaches 0 as the number of rectangles approaches infinity. (In other words, $\|\Delta\| \to 0$ as $n \to \infty$). Then the double integral of *f* over *R* is

$$\iint_R f(x,y)\,dA = \lim_{\|\Delta\|\to 0, n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i .$$

where ΔA_i is the area of the *i*th rectangle, and (x_i, y_i) is any point in the *i*th rectangle (provided this limit exists). If this limit exists, then *f* is *integrable* over *R*.

Volume of a Solid Region

If *f* is integrable over a plane region *R* and $f(x,y) \geq 0$ for all (x,y) in *R*, then the volume of the solid region that lies above *R* and below the graph of *f* is

$$\iint_R f(x,y)\,dA .$$

Properties of Double Integrals

1. $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$

2. $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

3. $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$

4. $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ (for $(x,y) \in R$)

5. $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.

To evaluate a double integral $\iint_R f(x,y)\,dA$, we must rewrite it as an iterated integral. (We replace dA by either $dy\,dx$ or by $dx\,dy$, and we replace R by the corresponding limits of integration.

<u>Fubini's Theorem</u>:

Suppose $f(x,y)$ is continuous on the plane region *R*.

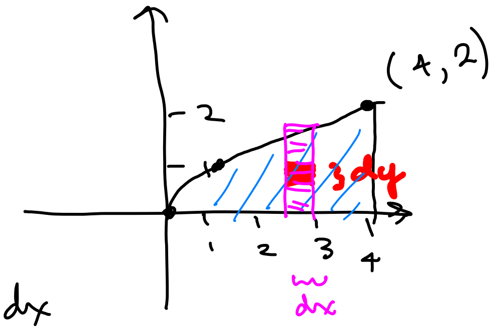
If *R* is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

R is defined by $c \le y \le d$ and $g_1(x) \le y \le g_2(x)$, where h_1 and h_2 are continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example 1: Evaluate $\iint_R \frac{y}{1+x^2}\,dA$, where R is the region bounded by the graphs of $y=0$, $y=\sqrt{x}$, and $x=4$.



$$\iint_R \frac{y}{1+x^2}\,dA = \int_0^4\int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx$$

$$= \int_0^4 \frac{1}{1+x^2}\int_0^{\sqrt{x}} y\,dy\,dx = \int_0^4 \frac{1}{1+x^2}\cdot\frac{y^2}{2}\Big|_0^{\sqrt{x}}\,dx$$

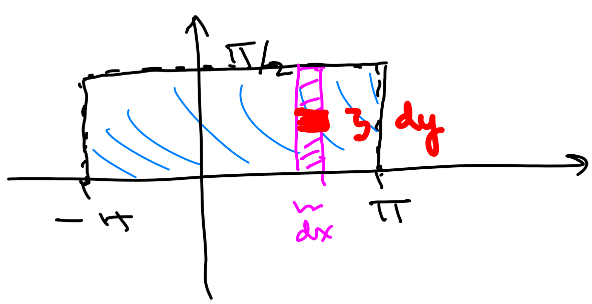
$$= \int_0^4 \frac{1}{1+x^2}\left(\frac{(\sqrt{x})^2}{2}-\frac{0^2}{2}\right)dx = \frac{1}{2}\int_0^4 \frac{x}{1+x^2}\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_1^{17}\frac{1}{u}\,du = \frac{1}{4}\ln|u|\Big|_1^{17}$$

$$= \frac{1}{4}\left(\ln 17 - \cancel{\ln 1}^{\,0}\right) = \boxed{\frac{1}{4}\ln 17}$$

$u = 1+x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$
$x=0 \Rightarrow u = 1+0^2 = 1$
$x=4 \Rightarrow u = 1+4^2 = 17$

Example 2: Evaluate $\iint_R \sin x \sin y\,dA$, where R is the rectangle with vertices $(-\pi,0)$, $(\pi,0)$, $(\pi,\frac{\pi}{2})$, and $(-\pi,\frac{\pi}{2})$.



$$\iint_R \sin x \sin y\,dA$$

$$= \int_{-\pi}^{\pi}\int_0^{\pi/2} \sin x \sin y\,dy\,dx$$

$$= \int_{-\pi}^{\pi} \sin x\,(-\cos y)\Big|_0^{\pi/2}\,dx = \int_{-\pi}^{\pi}\sin x\left(-\cos\frac{\pi}{2} - (-\cos 0)\right)dx$$

$$= \int_{-\pi}^{\pi}\sin x\,(-0+1)\,dx = \int_{-\pi}^{\pi}\sin x\,dx = -\cos x\Big|_{-\pi}^{\pi}$$

$$= -\cos\pi + \cos(-\pi)$$

$$= -(-1) + (-1) = 1 - 1 = \boxed{0}$$

Example 3: Evaluate $\iint_R (2x+3y+7)\,dA$, where R is the region bounded by the graphs of $y=\frac{1}{3}x$ and $y=\sqrt{x}$.

Example 4: Find the volume of the solid bounded above by the surface $x+2y+3z=6$ in the first octant.

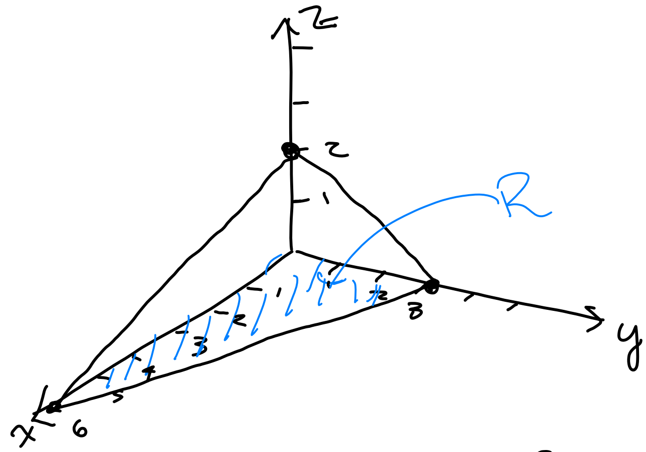
Traces:

xy-plane: $z=0 \Rightarrow x+2y=6$

yz-plane: $x=0 \Rightarrow 2y+3z=6$

xz plane: $y=0 \Rightarrow x+3z=6$

$x+2y=6$	$2y+3z=6$
$x=0 \Rightarrow y=3$	$y=0 \Rightarrow z=2$
$y=0 \Rightarrow x=6$	$z=0 \Rightarrow y=3$



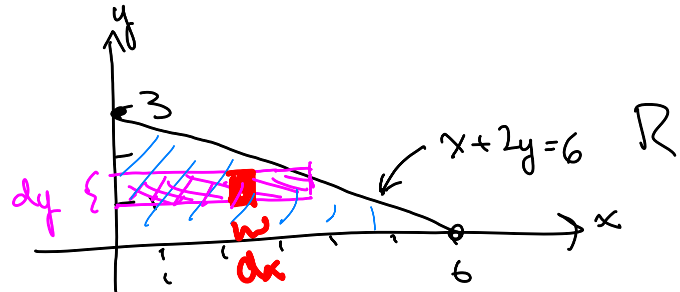
write as $z=f(x,y)$: $x+2y+3z=6$

$3z=6-x-2y$

$z=\frac{1}{3}(6-x-2y)$

See next page

$$V = \iint_R \frac{1}{3}(6 - x - 2y)\, dA$$



$$= \frac{1}{3}\int_0^3 \int_0^{6-2y} (6 - x - 2y)\, dx\, dy$$

$$= \frac{1}{3}\int_0^3 \left[6x - \frac{x^2}{2} - 2yx \right]\Bigg|_0^{6-2y} dy$$

$$x + 2y = 6$$

$$2y = -x + 6$$

$$y = -\frac{1}{2}x + 3$$

$$x = 6 - 2y$$

$$= \frac{1}{3}\int_0^3 \left[6(6-2y) - \frac{(6-2y)^2}{2} - 2y(6-2y) \right] dy$$

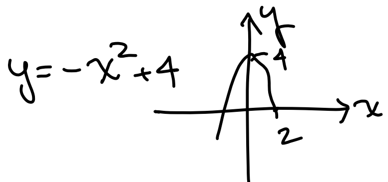
$$= \frac{1}{3}\int_0^3 \left[36 - 12y - \frac{36 - 24y + 4y^2}{2} - 12y + 4y^2 \right] dy$$

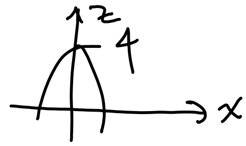
$$= \frac{1}{3}\int_0^3 \left[36 - \cancel{12y} - 18 + \cancel{12y} - 2y^2 - 12y + 4y^2 \right] dy$$

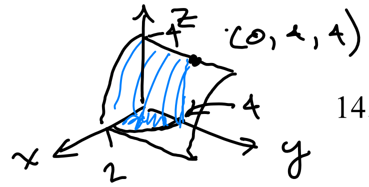
$$= \frac{1}{3}\int_0^3 \left[2y^2 - 12y + 18 \right] dy = \frac{1}{3} \left[\frac{2y^3}{3} - \frac{12y^2}{2} + 18y \right]\Bigg|_0^3$$

$$= \frac{1}{3}\left[\frac{2(3)^3}{3} - 6(3)^2 + 18(3) - 0 \right]$$

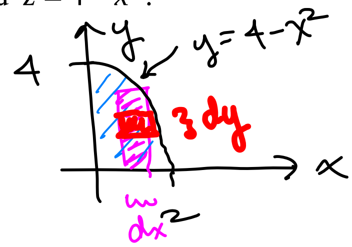
$$= \frac{1}{3}\left[18 - 54 + 54 \right] = \boxed{6}$$

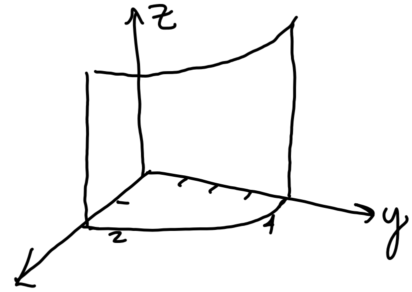






Example 5: Find the volume of the solid in the first octant bounded by the graphs of $y = 4 - x^2$ and $z = 4 - x^2$.





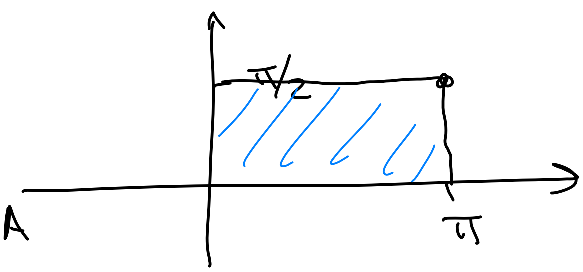
$$V = \int_0^2 \int_0^{4-x^2} (4-x^2)\,dy\,dx = \int_0^2 \left(4y - x^2 y\right)\Big|_0^{4-x^2} dx = \int_0^2 \left[4(4-x^2) - x^2(4-x^2)\right] dx$$

$$\ldots \frac{256}{15}$$

Average value of a function:

> Definition: Average Value of a Function
>
> If f is integrable over a plane region R, then the average value of f over R is
>
> $$\text{Average value} = \frac{1}{A}\iint_R f(x,y)\,dA,$$
>
> where A is the area of R.

Example 6: Find the average value of $f(x, y) = \sin(x + y)$ on the rectangle with vertices $(0,0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$ and $(0, \frac{\pi}{2})$.



$$\text{Area} = \pi\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2}$$

$$\text{Avg value} = \frac{1}{A}\iint_R \sin(x+y)\,dA$$

$$= \frac{2}{\pi^2}\int_0^{\pi/2}\int_0^{\pi} \sin(x+y)\,dx\,dy$$

works out to $\boxed{\frac{4}{\pi^2}}$

$$= \frac{2}{\pi^2}\int_0^{\pi/2} \left[-\cos(x+y)\right]\Big|_0^{\pi} dy$$

$$= \frac{2}{\pi^2}\int_0^{\pi/2} \left[-\cos(\pi+y) + \cos(y)\right] dy = \frac{2}{\pi^2}\left[-\sin(\pi+y) + \sin(y)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi^2}\left[-\sin(\pi + \pi/2) + \sin(\pi/2) + \sin(\pi+0) - \sin(0)\right]$$

$$= \frac{2}{\pi^2}\left[-\sin\left(\frac{3\pi}{2}\right) + 1 + 0 - 0\right] = \frac{2}{\pi^2}\left[-(-1) + 1\right] = \frac{2}{\pi^2}(2)$$

$$= \boxed{4/\pi^2}$$