14.3: Change of Variables: Polar Coordinates

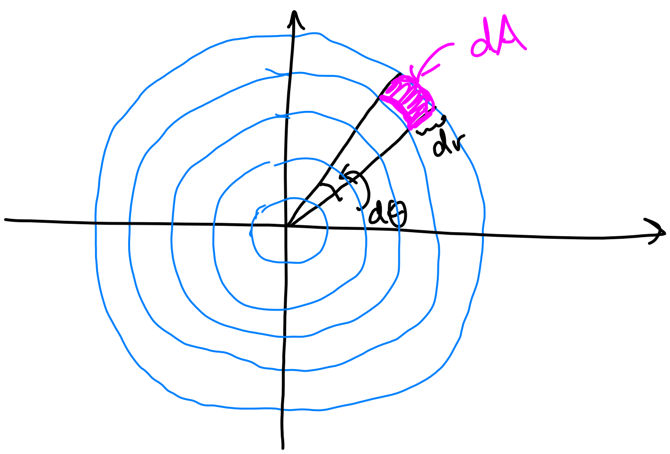
For some regions, using polar coordinates makes sense (and makes the integration easier)!

Recall: To convert between polar and rectangular coordinates, use the following equations:

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$$

For regions that are variations of circles, we can write dA in terms of r and θ:



It can be shown that $\boxed{dA = r\,dr\,d\theta}$ (see Summer 2015 notes)

Theorem: Changing a Double Integral to Polar Form

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta .$$

Example 1: Use a double integral to calculate the area of a circle of radius 3.

$$\text{Area} = 4\iint_R dA = 4\int_0^3\int_0^{\sqrt{9-x^2}} dy\,dx = 4\int_0^3 y\Big|_0^{\sqrt{9-x^2}} dx$$

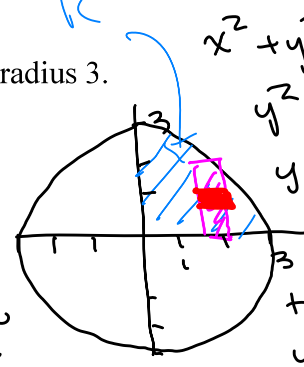
$$= 4\int_0^3 \sqrt{9-x^2}\,dx$$ use trig substitution to integrate

$x^2+y^2=9$

$y^2 = 9-x^2$

$y = \pm\sqrt{9-x^2}$

top half: $y=\sqrt{9-x^2}$

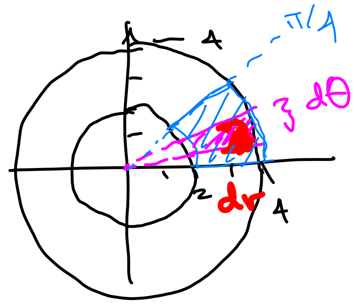


In polar: $A = 4\iint_R dA = 4\int_0^{\pi/2}\int_0^3 r\,dr\,d\theta$

$$= 4\int_0^{\pi/2} \frac{r^2}{2}\Big|_0^3 d\theta = 4\int_0^{\pi/2}\left(\frac{9}{2}-0\right)d\theta$$

$$= 4\left(\frac{9}{2}\right)\theta\Big|_0^{\pi/2} = \frac{36}{2}\left(\frac{\pi}{2}-0\right) = \boxed{9\pi}$$

Example 2: For $\int_0^{\pi/4}\int_2^4 r^2\cos\theta\sin\theta\,dr\,d\theta$, sketch the region of integration and calculate the iterated integral.



$$\int_0^{\pi/4}\int_2^4 r^2\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4}\cos\theta\sin\theta\cdot\frac{r^3}{3}\Big|_2^4 d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4}\cos\theta\sin\theta\left[4^3-2^3\right]d\theta = \frac{56}{3}\int_0^{\pi/4}\cos\theta\sin\theta$$

Example 3: Calculate $\iint_R (x+y)\,dA$, where R is the disk bounded by $x^2+y^2=25$.

$$\iint_R (x+y)\,dA = \int_0^{2\pi}\int_0^5 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

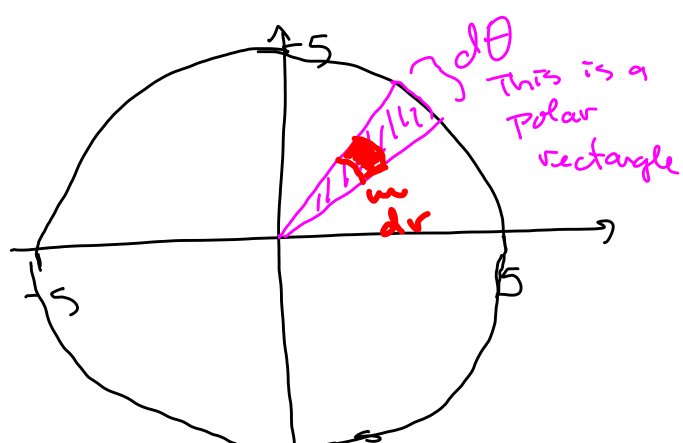
$$= \int_0^{2\pi}\int_0^5 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_0^5 d\theta$$

$$= \int_0^{2\pi}\left(\frac{5^3}{3}-\frac{0^3}{3}\right)(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{125}{3}\left[\sin\theta-\cos\theta\right]\Big|_0^{2\pi} = \frac{125}{3}\left[\sin 2\pi - \cos 2\pi - \sin 0 + \cos 0\right]$$

$$= \frac{125}{3}\left[0-1-0+1\right] = \boxed{0}$$



Example 4: Evaluate $\int_0^2\int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\,dy$ by converting to polar coordinates.

$$\int_0^2\int_y^{\sqrt{8-y^2}}\sqrt{x^2+y^2}\;dx\;dy$$

$$=\iint_R \sqrt{x^2+y^2}\;dA$$

$\sqrt{x^2+y^2}=r$

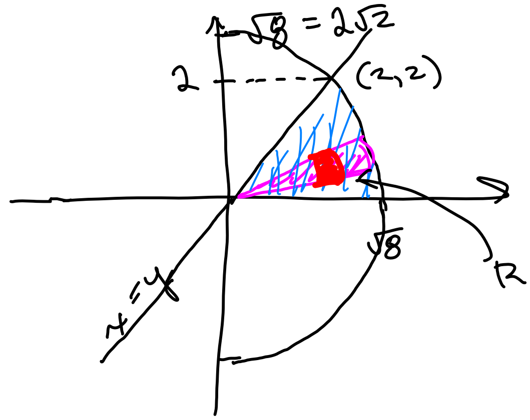
$x^2+y^2=r^2$

$$=\int_0^{\pi/4}\int_0^{\sqrt8} r\;r\,dr\,d\theta = \int_0^{\pi/4}\int_0^{\sqrt8} r^2\,dr\,d\theta$$

$$=\int_0^{\pi/4}\frac{r^3}{3}\Big|_0^{\sqrt8}d\theta = \int_0^{\pi/4}\left[\frac{\sqrt8\sqrt8\sqrt8}{3}-\frac{0^3}{3}\right]d\theta$$

$$=\frac{8\sqrt8}{3}\theta\Big|_0^{\pi/4} = \frac{\cancel{8}^{\,2}\cdot 2\sqrt2}{3}\left[\frac{\pi}{\cancel{4}}-0\right]$$

$$=\boxed{\frac{4\pi\sqrt2}{3}}$$



dx runs from $x=y$ to $x=\sqrt{8-y^2}$

$\hookrightarrow x^2=8-y^2$

$x^2+y^2=8$

Right half of circle of radius $\sqrt8$

Where do circle and line intersect?

$y=\sqrt{8-y^2}$

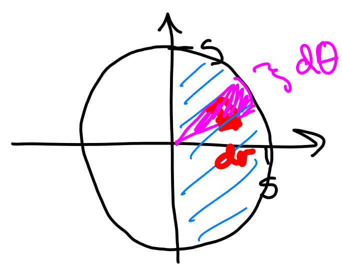
$y^2=8-y^2 \Rightarrow 2y^2=8$

$y^2=4$

$y=\pm2$

Example 5: Calculate $\iint_R f(x,y)\,dA$, where R is the region described by $x^2+y^2\le 25$, $x\ge 0$, and $f(x,y)=e^{-(x^2+y^2)/2}$.

$$\iint_R e^{-\frac{x^2+y^2}{2}}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^5 e^{-\frac{r^2}{2}}\,r\,dr\,d\theta$$



$$=-\int_{-\pi/2}^{\pi/2}\int_0^{-25/2} e^u\,du\;d\theta$$

$u=-\frac{1}{2}r^2$

$du=-r\,dr$

$-du=r\,dr$

$r=0 \Rightarrow u=-\frac{1}{2}(0)^2=0$

$r=5 \Rightarrow u=-\frac{1}{2}(5)^2=-\frac{25}{2}$

$$=-\int_{-\pi/2}^{\pi/2} e^u\Big|_0^{-25/2}d\theta = -\int_{-\pi/2}^{\pi/2}\left[e^{-25/2}-e^0\right]d\theta$$

$$=-\left[e^{-25/2}-1\right]\theta\Big|_{-\pi/2}^{\pi/2} = -\left(e^{-25/2}-1\right)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)$$

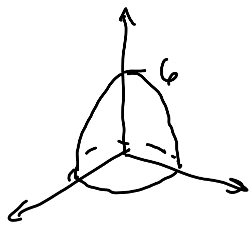
$$=\left(1-e^{-25/2}\right)(\pi) = \boxed{\pi-\pi e^{-25/2}} = \pi-\frac{\pi}{e^{12.5}}$$

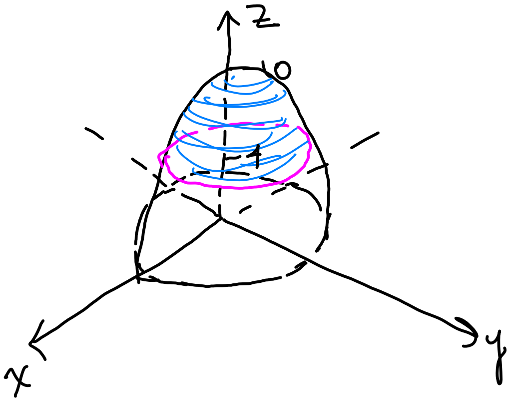
Example 6: Calculate $\iint\limits_R f(x,y)\,dA$, where R is the region described by $x^2 + y^2 \le 9$, $x \ge 0$, $y \ge 0$ and $f(x,y) = 9 - x^2 - y^2$.

Example 7: Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

Example 8: Find the volume of the solid bounded by the paraboloid $z=10-3x^2-3y^2$ and the plane $z=4$.

Modify the function by subtracting 4 from all the z's.





$$z = 10-3x^2-3y^2-4$$

$$z=6-3x^2-3y^2$$

where does the new paraboloid intersect the xy-plane?

Set $z=0$: $0=6-3x^2-3y^2$

$$3x^2+3y^2=6$$

$$x^2+y^2=2 \quad \text{circle of radius } \sqrt{2}$$

$x=0, y=0$

$\Downarrow$

$z=10$ max value of z

Traces:

$x=0 \Rightarrow z=10-3y^2$ parabola in yz-plane

xz-plane: $y=0 \Rightarrow z=10-3x^2$ parabola

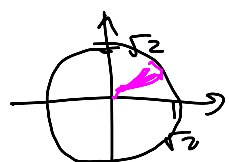
xy-plane $\Rightarrow z=0 \Rightarrow$

$$0=10-3x^2-3y^2$$

$$3x^2+3y^2=10$$

$$x^2+y^2=\frac{10}{3}$$

$$V=\int_0^{2\pi}\int_0^{\sqrt{2}}\left(6-3(x^2+y^2)\right) r\,dr\,d\theta = \boxed{\int_0^{2\pi}\int_0^{\sqrt{2}}(6-3r^2)\,r\,dr\,d\theta}$$



See summer notes for calculation

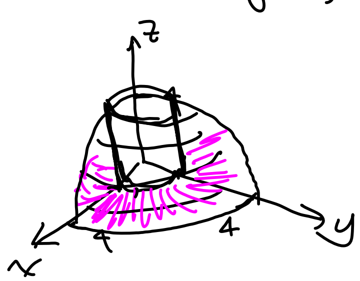
Example 9: Find the volume of the solid inside the hemisphere $z=\sqrt{16-x^2-y^2}$ and outside the cylinder $x^2+y^2=1$.

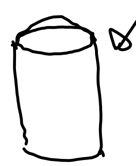
For any z, trace is a circle of radius 1

$$z^2=16-x^2-y^2$$

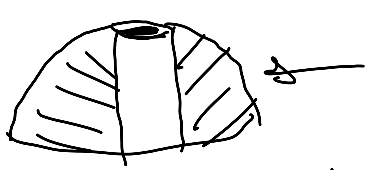
$x^2+y^2+z^2=16$ sphere radius 4

$z=\sqrt{16-x^2-y^2}$ is the top half



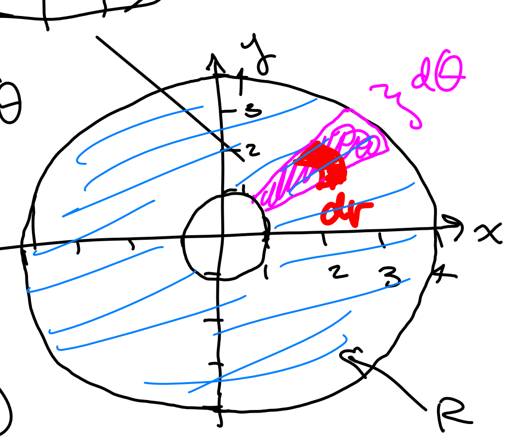


cut out part



the part we want

$$V=\iint_R \sqrt{16-x^2-y^2}\,dA = \int_0^{2\pi}\int_1^4 \sqrt{16-r^2}\,r\,dr\,d\theta$$



$$= -\frac{1}{2}\int_0^{2\pi}\int_{15}^{0} u^{1/2}\,du\,d\theta = -\frac{1}{2}\int_0^{2\pi} \left.\frac{u^{3/2}}{3/2}\right|_{15}^{0} d\theta$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[0^{3/2}-15^{3/2}\right]\theta\Big|_0^{2\pi} = -\frac{1}{3}\left(-15^{3/2}\right)(2\pi-0)$$

$$= \frac{2\pi(15)^{3/2}}{3} = \frac{2\pi\cdot 15\sqrt{15}}{3}$$

$$\boxed{10\pi\sqrt{15}}$$

$u=16-r^2$ $\quad$ $r=1 \Rightarrow u=15$

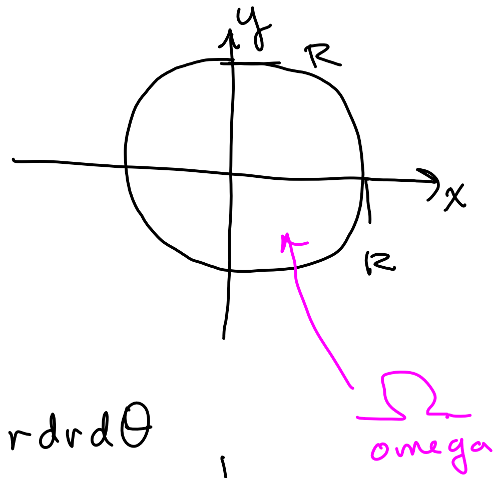
$du=-2r\,dr$ $\quad$ $r=4 \Rightarrow u=0$

$-\frac{1}{2}du = r\,dr$

Example 10: Find the area of the region inside the graph of $r = 2\cos\theta$ and outside the graph of $r = 1$.

Example 11: Find the area of the region enclosed by the graph of $r = 2 + \sin\theta$.

Example 12: Find the volume of the sphere of radius *R*.



Eqn of Sphere: $x^2+y^2+z^2=R^2$

$$z=\pm\sqrt{R^2-x^2-y^2}$$

Top half: $z=\sqrt{R^2-x^2-y^2}$

Volume of whole sphere:

$$V=2\iint_{\Omega}\sqrt{R^2-x^2-y^2}\,dA=2\int_0^{2\pi}\int_0^R\sqrt{R^2-r^2}\;r\,dr\,d\theta$$

$u=R^2-r^2$

$du=-2r\,dr$

$-\frac{1}{2}du=r\,dr$

$r=0\Rightarrow u=R^2-0^2=R^2$

$r=R\Rightarrow u=R^2-R^2=0$

$$=2\left(-\frac{1}{2}\right)\int_0^{2\pi}\int_{R^2}^{0}u^{1/2}\,du\,d\theta$$

$$=-\int_0^{2\pi}\left.\frac{u^{3/2}}{3/2}\right|_{R^2}^{0}d\theta=-\frac{2}{3}\int_0^{2\pi}\left[0^{3/2}-\left(R^2\right)^{3/2}\right]d\theta$$

$$=\frac{2}{3}R^3\int_0^{2\pi}d\theta=\frac{2}{3}R^3\theta\Big|_0^{2\pi}$$

$$=\frac{2}{3}R^3(2\pi-0)=\boxed{\frac{4\pi}{3}R^3}=\boxed{\frac{4}{3}\pi R^3}$$