14.5: Surface Area

We can use a double integral to find the upper surface area of a solid defined by $z = f(x, y)$ over a region *R*.

> Definition: If *f* and its first partial derivatives are continuous on the closed region *R* in the *xy*-plane, then the area of the surface S given by $z = f(x, y)$ over R is defined as
>
> $$\text{Surface Area} = \iint_R dS = \iint_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA.$$

Note:

Length on *x*-axis: $\int_a^b dx$ Length $= \int_a^b dx = x\Big|_a^b = b-a$

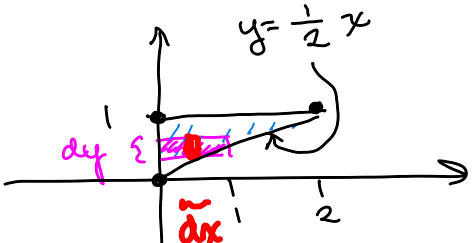
Arc length in *xy*-plane: $\int_a^b \sqrt{1+[f'(x)]^2}\, dx$ $\int_a^b \sqrt{1+[f'(x)]^2}\, dx$

Area in *xy*-plane: $\iint_R dA$

Surface area in $\mathbb{R}^3$: $\iint_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$

Example 1: Find the area of the surface $z = 1+3x+2y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$, and $(2,1)$.

$z = f(x,y) = 1+3x+2y^2$



equ of line: $y = \frac{1}{2}x$

$2y = x$

$$\text{Surface Area} = \iint_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\, dA$$

$$= \int_0^1\int_0^{2y} \sqrt{1+(3)^2+(4y)^2}\, dx\, dy$$

$$= \int_0^1 \sqrt{1+9+16y^2}\int_0^{2y} dx\, dy = \int_0^1 \left(\sqrt{10+16y^2}\right) x\Big|_0^{2y} dy$$

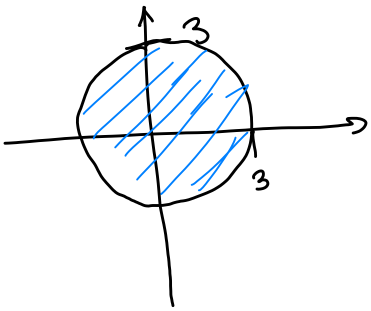
$$= \int_0^1 \sqrt{10+16y^2}\,(2y-0)\, dy = \int_0^1 2y\sqrt{10+16y^2}\, dy = 2\int_0^1 y\,(10+16y^2)^{1/2}\, dy$$

$$= 2\left(\frac{1}{32}\right)\frac{(10+16y^2)^{3/2}}{3/2}\Bigg|_0^1 = \frac{1}{16}\cdot\frac{2}{3}\left[(10+16)^{3/2}-(10+0)^{3/2}\right]$$

$$= \boxed{\frac{1}{24}\left[26^{3/2}-10^{3/2}\right]} \approx 4.2063$$

Note: if your integrand can be written as $h(r)g(\theta)$ [or $h(x)g(y)$], then
$$\int_a^b\int_c^d h(r)\,g(\theta)\,dr\,d\theta = \int_a^b g(\theta)\,d\theta \int_c^d h(r)\,dr \quad \left[\text{or } \int_a^b\int_c^d h(x)g(y)\,dx\,dy = \int_c^d h(x)\,dx \int_a^b g(y)\,dy\right]$$

Example 2: Find the area of the surface $f(x,y) = 12 + 2x - 3y$ that lies above the region R bounded by the graph of $x^2 + y^2 \le 9$.



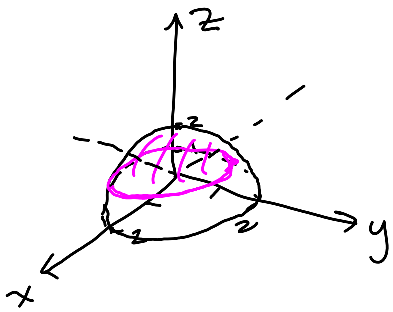
$$S = \text{Surface Area} = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\; dA$$

$$= \iint_R \sqrt{1 + (2)^2 + (-3)^2}\; dA = \sqrt{14}\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$$

$$= \sqrt{14}\; \theta\Big|_0^{2\pi} \cdot \frac{r^2}{2}\Big|_0^3 = \sqrt{14}\,(2\pi - 0)\left(\frac{3^2}{2} - \frac{0^2}{2}\right) = \sqrt{14}\,(2\pi)\left(\frac{9}{2}\right)$$

$$= \boxed{9\pi\sqrt{14}}$$

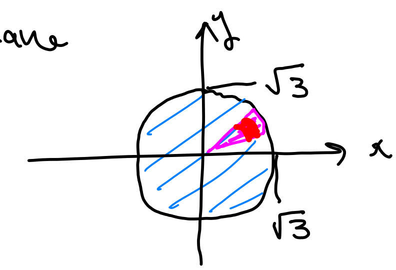
Example 3: Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.



$$z^2 = 4 - x^2 - y^2$$
$$z = \pm\sqrt{4 - x^2 - y^2}$$

Top half of sphere: $z = \sqrt{4 - x^2 - y^2}$

Project the part that lies above the plane $z = 1$ onto the xy-plane



Set $z = 1$: $x^2 + y^2 + 1^2 = 4$
$$x^2 + y^2 = 3$$
Circle of radius $\sqrt{3}$

For $z = \sqrt{4 - x^2 - y^2} = (4 - x^2 - y^2)^{1/2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(4 - x^2 - y^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4 - x^2 - y^2}} \Rightarrow \left[\frac{\partial z}{\partial x}\right]^2 = \frac{x^2}{4 - x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(4 - x^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{4 - x^2 - y^2}} \Rightarrow \left[\frac{\partial z}{\partial y}\right]^2 = \frac{y^2}{4 - x^2 - y^2}$$

See next page

$$S = \iint_R \sqrt{1 + \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2}\; dA = \iint_R \sqrt{1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2}}\; dA$$

$$= \iint_R \sqrt{\frac{4 - x^2 - y^2 + x^2 + y^2}{4 - x^2 - y^2}}\; dA = \iint_R \sqrt{\frac{4}{4 - (x^2 + y^2)}}\; dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2}{\sqrt{4 - r^2}}\, r\,dr\,d\theta$$

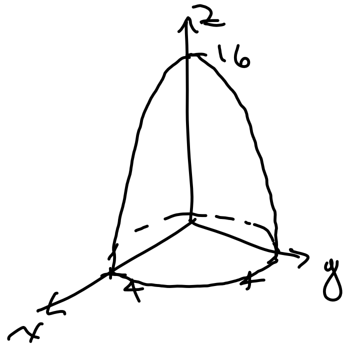
Ex 3 cont'd: $S = 2\int_0^{2\pi}\int_0^{\sqrt{3}} (4-r^2)^{-1/2} r \, dr \, d\theta = 2\left(\frac{1}{-2}\right)\int_0^{2\pi} \frac{(4-r^2)^{1/2}}{1/2}\Big|_0^{\sqrt{3}} d\theta$

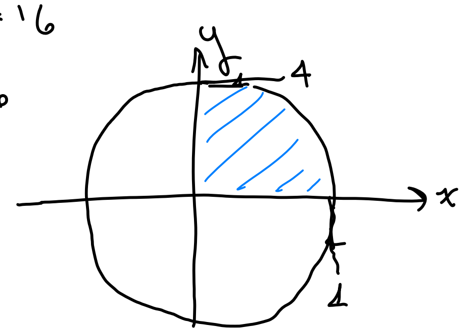
$= -2\left[(4-3)^{1/2} - (4-0)^{1/2}\right]\theta\Big|_0^{2\pi} = -2[1-2](2\pi - 0) = -2(-1)(2\pi) = \boxed{4\pi}$

Example 4: Find the surface area of the portion of the paraboloid $z = 16 - x^2 - y^2$ that lies in the first octant.

$z = 0 \Rightarrow \quad x^2 + y^2 = 16$

$x = 0, y = 0 \Rightarrow z = 16$





$$S = \iint_R \sqrt{1 + \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2} \, dA = \iint_R \sqrt{1 + (-2x)^2 + (-2y)^2} \, dA$$

$$= \iint_R \sqrt{1 + 4x^2 + 4y^2} \, dA = \iint \sqrt{1 + 4\underbrace{(x^2+y^2)}_{r^2}} \, dA$$

$$= \int_0^{\pi/2}\int_0^4 \sqrt{1+4r^2} \, r \, dr \, d\theta = \int_0^{\pi/2} d\theta \int_0^4 (1+4r^2)^{1/2} r \, dr$$

$$= \theta\Big|_0^{\pi/2} \cdot \frac{1}{8} \cdot \frac{(1+4r^2)^{3/2}}{3/2}\Big|_0^4$$

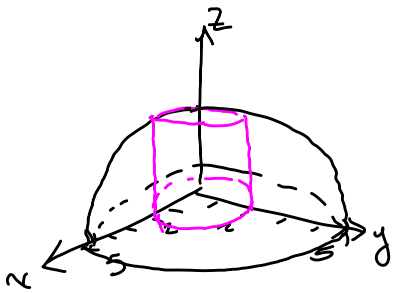
$$= \left(\frac{\pi}{2} - 0\right)\left(\frac{1}{8}\right)\left(\frac{2}{3}\right)\left[(1+4(4^2))^{3/2} - (1+4(0)^2)^{3/2}\right]$$

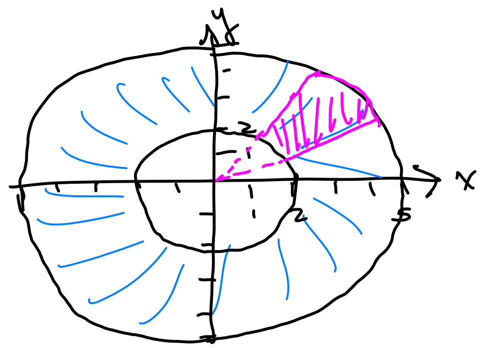
$$= \boxed{\frac{\pi}{24}\left[65^{3/2} - 1\right]}$$

Example 5: Find the surface area of the portion of the hemisphere $z = \sqrt{25 - x^2 - y^2}$ that lies outside the cylinder $x^2 + y^2 = 4$.

Sphere of radius 5.

Cylinder of radius 2





$$\frac{\partial z}{\partial x} = \frac{1}{2}(25 - x^2 - y^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{25-x^2-y^2}} \Rightarrow \frac{x^2}{25-x^2-y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(25 - x^2 - y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{25-x^2-y^2}} \Rightarrow \frac{y^2}{25-x^2-y^2}$$

$$S = \iint_R \sqrt{1 + \left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2}\, dA = \int_0^{2\pi}\int_2^5 \sqrt{1 + \frac{x^2}{25-x^2-y^2} + \frac{y^2}{25-x^2-y^2}}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^5 \sqrt{\frac{25-x^2-y^2 + x^2 + y^2}{25-x^2-y^2}}\; r\,dr\,d\theta = \int_0^{2\pi}\int_2^5 \sqrt{\frac{25}{25-r^2}}\; r\,dr\,d\theta$$

$$= 5\int_0^{2\pi}\int_2^5 (25 - r^2)^{-1/2}\, r\,dr\,d\theta$$

$\vdots$

turns out to equal $\boxed{10\pi\sqrt{21}}$