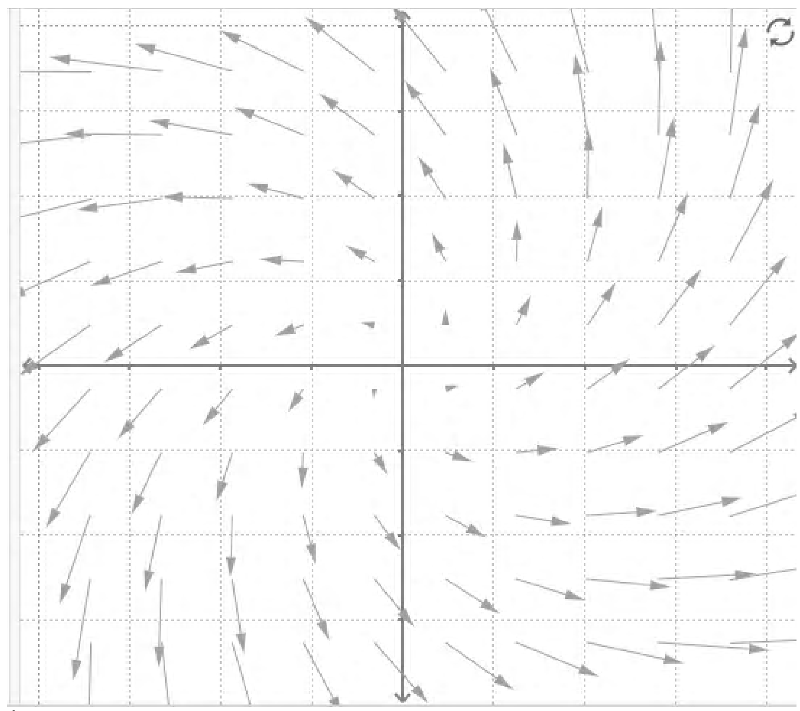
15.1: Vector Fields

Vector fields are functions that assign a vector to each point in $\mathbb{R}^2$ or $\mathbb{R}^3$.

We can represent a vector field by a mesh of arrows at regularly spaced intervals.

Example 1: This is a graphic representation of the vector field $\mathbf{F}(x,y)=\langle x-y, x+y\rangle$.



(x,y)	$\vec{F}(x,y)$
$(0,0)$	$\langle 0-0, 0+0\rangle = \langle 0,0\rangle$
$(1,0)$	$\langle 1-0, 1+0\rangle = \langle 1,1\rangle$
$(2,0)$	$\langle 2-0, 2+0\rangle = \langle 2,2\rangle$
$(0,1)$	$\langle 0-1, 0+1\rangle = \langle -1,1\rangle$
$(0,2)$	$\langle 0-2, 0+2\rangle = \langle -2,2\rangle$
$(1,1)$	$\langle 1-1, 1+1\rangle = \langle 0,2\rangle$
$(2,2)$	$\langle 2-2, 2+2\rangle = \langle 0,4\rangle$
$(-3,-1)$	$\langle -3-(-1), -3-1\rangle = \langle -2,-4\rangle$

Example 2: Plot a few vectors in the field for $\mathbf{F}(x,y)=\langle x^2y, xy^2\rangle$.

$\vec{F}(x,y) = \langle x^2y, xy^2\rangle$

$\vec{F}(0,0) = \langle 0,0\rangle$

$\vec{F}(1,0) = \langle 0,0\rangle$

$\vec{F}(1,1) = \langle 1,1\rangle$

$\vec{F}(2,1) = \langle 2^2(1), 2(1)^2\rangle = \langle 4,2\rangle$

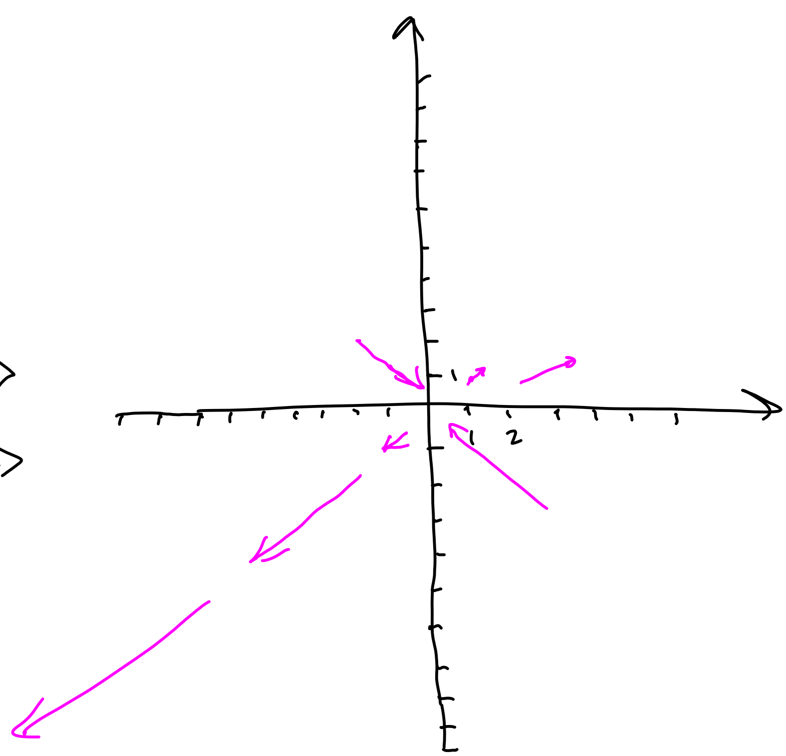
$\vec{F}(-2,2) = \langle (-2)^2(2), -2(2)^2\rangle = \langle 8,-8\rangle$

$\vec{F}(-1,-1) = \langle (-1)^2(-1), -1(-1)^2\rangle = \langle -1,-1\rangle$

$\vec{F}(-2,-2) = \langle -8,-8\rangle$

$\vec{F}(2,-2) = \langle 2^2(-2), 2(-2)^2\rangle = \langle -8, 8\rangle$

$\vec{F}(2,-1) = \langle 2^2(-1), 2(-1)^2\rangle = \langle -4, 2\rangle$



Note: A gradient function $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$, or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$, is a vector field. Physical examples of vector fields include velocity fields, gravitational fields, and electric force fields.

Conservative vector fields and potential functions:

Definition: A vector field **F** is called *conservative* if there exists a differentiable function f such that $\mathbf{F} = \nabla f$. The function f is called the *potential function* for **F**.

Example 3: Find a conservative vector field for the potential function $f(x,y) = x^3y^2 + 2xy^4$.

Conservative vector field is $\vec{F}(x,y) = \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$$= \langle 3x^2y^2 + 2y^4, 2x^3y + 8xy^3 \rangle$$

$$\vec{F}(x,y) = (3x^2y^2 + 2y^4)\hat{\imath} + (2x^3y + 8xy^3)\hat{\jmath}$$

Theorem: Test for a Conservative Vector Field (in $\mathbb{R}^2$)

Let functions M and N have continuous first partial derivatives on an open disk R. The vector field $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Why is this true?

Suppose $\vec{F}(x,y) = \nabla f(x,y)$, where $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ have continuous 1st partials.

Then $f_x = M$ and $f_y = N$

Mixed partials: $f_{xy} = \frac{\partial M}{\partial y}$ and $f_{yx} = \frac{\partial N}{\partial x}$

Mixed partials are equal if 1st partials are continuous, so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 4: Determine whether the vector field $\mathbf{F}(x,y)=(x^2y^3-2x)\mathbf{i}+(3x^4y-2y)\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

$M(x,y) = x^2y^3 - 2x$ $\qquad \dfrac{\partial M}{\partial y} = 3x^2y^2$

$N(x,y) = 3x^4y - 2y$ $\qquad \dfrac{\partial N}{\partial x} = 12x^3y$

No, $\vec{F}$ is not conservative because $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Example 5: Determine whether the vector field $\mathbf{F}(x,y)=(x^3y^2-4x)\mathbf{i}+(\frac{1}{2}x^4y-6y)\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

$M = x^3y^2 - 4x$, so $\dfrac{\partial M}{\partial y} = 2x^3y$

$N = \frac{1}{2}x^4y - 6y$, so $\dfrac{\partial N}{\partial x} = \frac{1}{2}(4x^3)y = 2x^3y$. Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, so $\vec{F}$ is conservative.

$\nabla f(x,y) = \langle M, N\rangle = \langle x^3y^2 - 4x, \frac{1}{2}x^4y - 6y\rangle$

$f_x = M \Rightarrow f(x,y) = \int f_x(x,y)\,dx = \int M\,dx = \int (x^3y^2 - 4x)\,dx = \frac{x^4}{4}\cdot y^2 - \frac{4x^2}{2} + g(y)$

$= \frac{1}{4}x^4y^2 - 2x^2 + g(y)$

$f_y = N \Rightarrow f(x,y) = \int f_y(x,y)\,dy = \int N\,dy = \int (\frac{1}{2}x^4y - 6y)\,dy = \frac{1}{2}x^4\cdot\frac{y^2}{2} - \frac{6y^2}{2} + h(x)$

$= \frac{1}{4}x^4y^2 - 3y^2 + h(x)$

These 2 expressions for f must be equal.

So, $g(y) = -3y^2$ and $h(x) = -2x^2$. Thus $\boxed{f(x,y) = \frac{1}{4}x^4y^2 - 3y^2 - 2x^2}$

Check that $\nabla f = \vec{F}$.

Example 6: Determine whether the vector field $\mathbf{F}(x,y)=3x^2y^2\mathbf{i}+2x^3y\mathbf{j}$ is conservative. If it is, find a potential function for the vector field.

$\vec{F}(x,y) = 3x^2y^2\,\hat{i} + 2x^3y\,\hat{j}$

$M = 3x^2y^2$, $N = 2x^3y$

$\dfrac{\partial M}{\partial y} = 6x^2y$, $\dfrac{\partial N}{\partial x} = 6x^2y$. So yes, $\vec{F}$ is conservative.

$f(x,y) = \int f_x(x,y)\,dx = \int M\,dx = \int 3x^2y^2\,dx = \frac{3x^3}{3}\cdot y^2 + g(y) = x^3y^2 + g(y)$

$f(x,y) = \int f_y(x,y)\,dy = \int N\,dy = \int 2x^3y\,dy = 2x^3\cdot\frac{y^2}{2} + h(x) = x^3y^2 + h(x)$

So, $\boxed{f(x,y) = x^3y^2 + K}$, where K is a constant.

Check: $\nabla f(x,y) = \langle 3x^2y^2, 2x^3y\rangle = \vec{F}(x,y)$

Curl of a vector field:

Definition: Curl

The *curl* of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\begin{aligned}\operatorname{curl}\mathbf{F}(x,y,z) &= \nabla \times \mathbf{F}(x,y,z) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.\end{aligned}$$

If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is said to be *irrotational*.

Example 7: Find the curl of $\mathbf{F}(x,y,z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$ at the point $(2,-1,3)$.

$\vec{F}(x,y,z) = x^2z\hat{i} - 2xz\hat{j} + yz\hat{k}$

$$\operatorname{curl}\vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2z & -2xz & yz \end{vmatrix} = \langle z-(-2x),\, x^2-0,\, -2z-0 \rangle$$

$$= \langle z+2x,\, x^2,\, -2z \rangle$$

$$\operatorname{curl}\vec{F}(2,-1,3) = \langle 3+2(2),\, 2^2,\, -2(3) \rangle = \boxed{\langle 7, 4, -6 \rangle}$$

Theorem: Test for a Conservative Vector Field (in $\mathbb{R}^3$)

Let functions M, N, and P have continuous first partial derivatives on an open sphere Q in $\mathbb{R}^3$. The vector field $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative if and only if

$$\operatorname{curl}\mathbf{F}(x,y,z) = \mathbf{0}. \leftarrow \vec{0} = \langle 0,0,0 \rangle$$

Consequently, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\ \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z},\ \text{and}\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Example 8: Determine whether the vector field $\mathbf{F}(x,y,z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3\rangle$$

$= \langle 0, 0, 0\rangle$. So $\vec{F}$ is conservative.

$$f(x,y,z) = \int f_x(x,y,z)\,dx = \int y^2z^3\,dx = y^2z^3x + g(y,z)$$

$$f(x,y,z) = \int f_y(x,y,z)\,dy = \int 2xyz^3\,dy = 2x\cdot\frac{y^2}{2}\cdot z^3 + h(x,z) = xy^2z^3 + h(x,z)$$

$$f(x,y,z) = \int f_z(x,y,z)\,dz = \int 3xy^2z^2\,dz = 3xy^2\cdot\frac{z^3}{3} + m(x,y) = xy^2z^3 + m(x,y)$$

$$\therefore \boxed{f(x,y,z) = xy^2z^3 + K, \text{ where } K \in \mathbb{R}}$$

Example 9: Determine whether the vector field $\mathbf{F}(x,y,z) = xyz\mathbf{i} - y^2\mathbf{j} + xz\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & -y^2 & xz \end{vmatrix} = \langle 0-0,\ xy - z,\ 0 - xz\rangle = \langle 0, xy-z, -xz\rangle$$

$$\neq \vec{0} = \langle 0,0,0\rangle$$

No, $\vec{F}$ is not conservative.

Example 10: Determine whether the vector field $\mathbf{F}(x,y,z) = 2xy\mathbf{i} + (x^2+z^2)\mathbf{j} + 2yz\mathbf{k}$ is conservative. If it is, find a potential function for the vector field.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+z^2 & 2yz \end{vmatrix} = \langle 2z - 2z,\ 0-0,\ 2x - 2x\rangle$$

$= \langle 0,0,0\rangle$. So yes, $\vec{F}$ is conservative.

$$f(x,y,z) = \int f_x\,dx = \int 2xy\,dx = \frac{2x^2}{2}y + g(y,z) = x^2y + g(y,z)$$

$$f(x,y,z) = \int f_y\,dy = \int (x^2+z^2)\,dy = x^2y + z^2y + h(x,z)$$

$$f(x,y,z) = \int f_z\,dz = \int 2yz\,dz = 2y\cdot\frac{z^2}{2} + m(x,y) = yz^2 + m(x,y)$$

$$g(y,z) = z^2y + k$$

$$f(x,y,z) = x^2y + g(y,z) = x^2y + z^2y + K.$$ Check that it works in all 3 of these

$$\boxed{f(x,y,z) = x^2y + z^2y + K}$$
$K \in \mathbb{R}$

Check: $\nabla f(x,y,z) = \langle 2xy,\ x^2+z^2,\ 2zy\rangle$
$= \vec{F}(x,y,z)$ ✓ ok

Divergence of a vector field:

> <u>Definition</u>: <u>Divergence</u>
>
> The *divergence* of $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ is
>
> $$\operatorname{div}\mathbf{F}(x,y) = \nabla \cdot \mathbf{F}(x,y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$
>
> $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$
>
> $\nabla \cdot \vec{F}(x,y) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle M, N \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$
>
> The *divergence* of $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is
>
> scalar
>
> $$\operatorname{div}\mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$
>
> If $\operatorname{div}\mathbf{F} = 0$, then $\mathbf{F}$ is said to be *divergence free*.

<u>Note</u>: $\operatorname{curl}\mathbf{F}$ is a vector function; $\operatorname{div}\mathbf{F}$ is a scalar function. Because $\operatorname{curl}\mathbf{F}$ is defined as a cross product, it does not make sense in $\mathbb{R}^2$.

<u>Example 11:</u> Find the divergence of $\mathbf{F}(x,y,z) = \ln(xyz)(\mathbf{i}+\mathbf{j}+\mathbf{k})$ at the point $(3,2,1)$.

$\vec{F}(x,y,z) = \langle \ln(xyz), \ln(xyz), \ln(xyz) \rangle$

$\operatorname{div}\vec{F}(x,y,z) = \frac{1}{xyz}(yz) + \frac{1}{xyz}(xz) + \frac{1}{xyz}(xy) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$\operatorname{div}\vec{F}(3,2,1) = \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{2}{6} + \frac{3}{6} + \frac{6}{6} = \boxed{\frac{11}{6}}$

<u>Example 12:</u> Given $\mathbf{F}(x,y,z) = e^{3x}y^2\mathbf{i} + 2xy^3z^4\mathbf{j} + y^3\sin(2z)\mathbf{k}$, find (a) $\operatorname{div}\mathbf{F}(x,y,z)$, (b) $\operatorname{div}\mathbf{F}(0,2,0)$, and (c) $\operatorname{div}(\operatorname{curl}\mathbf{F}(x,y,z))$.

$\vec{F}(x,y,z) = \langle e^{3x}y^2, 2xy^3z^4, y^3\sin(2z) \rangle$

(a) $\operatorname{div}\vec{F}(x,y,z) = 3e^{3x}y^2 + 6xy^2z^4 + 2y^3\cos(2z)$

(b) $\operatorname{div}\vec{F}(0,2,0) = 3e^0(2)^2 + 6(0) + 2(2)^3\cos(0) = 12 + 0 + 16(1) = \boxed{28}$

(c) $\operatorname{curl}\vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{3x}y^2 & 2xy^3z^4 & y^3\sin(2z) \end{vmatrix} = \langle 3y^2\sin(2z) - 8xy^3z^3, 0-0, 2y^3z^4 - 2e^{3x}y \rangle$

> <u>Theorem</u>:
>
> If $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field and M, N, and P have continuous second partial derivatives, then
>
> $$\operatorname{div}(\operatorname{curl}\mathbf{F}(x,y,z)) = 0.$$

$\operatorname{div}(\operatorname{curl}\vec{F}(x,y,z)) = \operatorname{div}\langle 3y^2\sin(2z) - 8xy^3z^3, 0, 2y^3z^4 - 2e^{3x}y \rangle$

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$$= \frac{\partial}{\partial x}\left(3y^2 \sin 2z - 8xy^3 z^3\right) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(2y^3 z^4 - 2e^{3x} y\right)$$

$$= 0 - 8y^3 z^3 + 0 + 8y^3 z^3 - 0$$

$$= \boxed{0} \quad \text{(scalar)}$$