15.2: Line Integrals

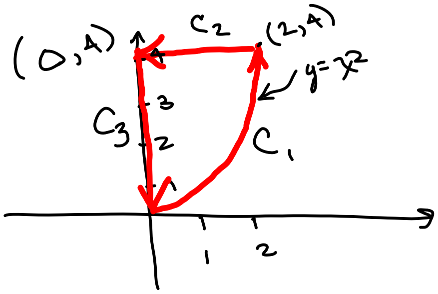
Recall: A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is considered *smooth* on the interval (a,b) if $x'(t)$, $y'(t)$, and $z'(t)$ are continuous on (a,b), and if $\mathbf{r}'(t) \neq \mathbf{0}$ for every t in (a,b). (In other words, if the curve is to be smooth, then $x'(t)$, $y'(t)$, and $z'(t)$ cannot be simultaneously 0 anywhere in the interval.)

Definition: A curve C is *piecewise smooth* on $[a,b]$ if the $[a,b]$ can be partitioned into a finite number of subintervals on which C is smooth.

Vector from $(2,4)$ to $(0,4)$ is $\langle -2,0\rangle$ direction vector

Could also use $\langle -1,0\rangle$

Example 1: Find a piecewise smooth parametrization of the path shown.



C_1: $\vec{r}_1(t) = \langle t, t^2\rangle$, $0 \le t \le 2$

C_2: $\vec{r}_2(t) = \langle -2t+6, 4\rangle$, $2 \le t \le 3$

C_3: $\vec{r}_3(t) = \langle 0,4\rangle + (t-3)\langle 0,-4\rangle$

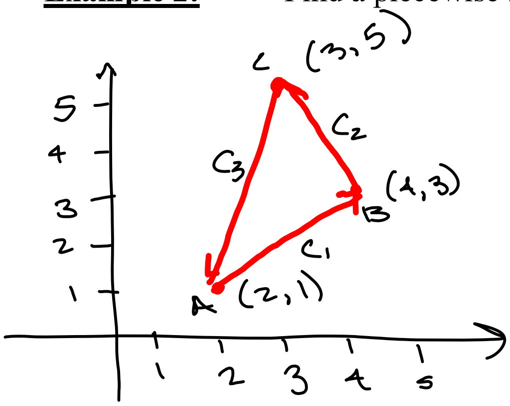
$= \langle 0,4\rangle + \langle 0, -4t+12\rangle$

$= \langle 0, -4t+16\rangle$, $3 \le t \le 4$

Direction vector for C_3:

$\langle 0,-4\rangle$

Example 2: Find a piecewise smooth parametrization of the path shown.



From A to B to C:

C_1: $\vec{r}_1(t) = \langle 2,1\rangle + t\langle 2,2\rangle$, $0 \le t \le 1$

$\langle 2+2t, 1+2t\rangle$, $t \in [0,1]$

C_2: $\vec{r}_2(t) = \langle 4,3\rangle + (t-1)\langle -1,2\rangle$

$= \langle 4,3\rangle + \langle -t+1, 2t-2\rangle$

$= \langle -t+5, 2t+1\rangle$, $1 \le t \le 2$

C_3: $\vec{r}_3(t) = \langle 3,5\rangle + (t-2)\langle -1,-4\rangle$

$= \langle 3,5\rangle + \langle -t+2, -4t+8\rangle$

$= \langle -t+5, -4t+13\rangle$, $2 \le t \le 3$

Line integrals:

Definition: Suppose f is defined in a region containing a smooth curve C of finite length. Suppose also that C is partitioned into n subarcs, with Δs_i representing the length of the ith subarc, and with $\|\Delta\|$ representing the length of the longest subarc. Then the line integral of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0,\, n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i\,.$$

Theorem: Evaluating a Line Integral

Suppose f is continuous in a region containing a smooth curve C. Suppose also that C is described by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ on an interval $[a,b]$, and that the curve is traversed exactly once as t increases from a to b. Then,

$$\int_C f(x,y,z)\,ds = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt$$
$$= \int_a^b f\big(x(t), y(t), z(t)\big)\|\mathbf{r}'(t)\|\,dt$$

$ds = \|\vec{r}\,'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Note: If $f(x,y,z) = 1$, then the line integral gives the arc length of C:

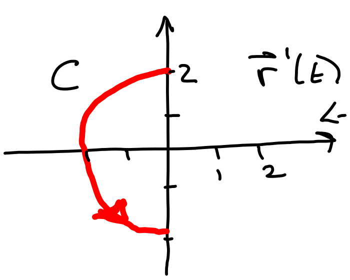
$$\text{Arc length} = \int_C 1\,ds = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\;dt = \int_a^b \|\mathbf{r}'(t)\|\,dt\,.$$

Note: The value of the line integral is independent of the parametrization chosen for C.

Example 3: Evaluate $\int_C (3x+4y)\,ds$, where C is the left half of the circle $x^2 + y^2 = 4$, traversed counterclockwise.

$y = \pm\sqrt{4-x^2}$

$x = \pm\sqrt{4-y^2}$



$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

C can be parametrized as

$x(t) = \cos t$, $y(t) = \sin t$

$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

or $x(t) = -\sqrt{4-t^2}$, $y(t) = -t$, $-2 \le t \le 2$

$$\int_C (3x+4y)\,ds = \int_{\pi/2}^{3\pi/2} (3\cdot 2\cos t + 4\cdot 2\sin t)\,\|\langle -2\sin t, 2\cos t\rangle\|\,dt$$

$$= \int_{\pi/2}^{3\pi/2} (6\cos t + 8\sin t)\sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt$$

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$$= \int_{\pi/2}^{3\pi/2} (6\cos t + 8\sin t)\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = \sqrt{4}\int_{\pi/2}^{3\pi/2} (6\cos t + 8\sin t)\, dt$$

$$= 2\left(6\sin t - 8\cos t\right)\Big|_{\pi/2}^{3\pi/2} = 12\sin\tfrac{3\pi}{2} - 16\cos\tfrac{3\pi}{2} - 12\sin\tfrac{\pi}{2} + 16\cos\tfrac{\pi}{2}$$

$$-12 - 0 - 12 + 0 = \boxed{-24}$$

Example 4: Evaluate $\int_C (x^2y + 3xy^3)\, ds$ along the line from (0,0) to (3,9).

$\vec{r}(t) = \langle 0,0\rangle + t\langle 3,9\rangle = \langle 3t, 9t\rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 3, 9\rangle$

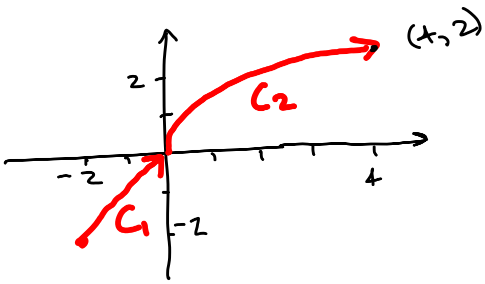
$\|\vec{r}\,'(t)\| = \sqrt{3^2+9^2} = \sqrt{9+81} = \sqrt{90}$

$$\int_C (x^2y + 3xy^3)\, ds = \int_0^1 \left((3t)^2(9t) + 3(3t)(9t)^3\right)\|\vec{r}\,'(t)\|\, dt$$

$$= \int_0^1 (81t^3 + 6561t^4)\sqrt{90}\, dt = \left(\frac{81t^4}{4} + 6561\frac{t^5}{5}\right)(3\sqrt{10})\Big|_0^1 = \boxed{3\sqrt{10}\left(\frac{81}{4} + \frac{6561}{5}\right)}$$

Example 5: Evaluate $\int_C (x^2 + y^2)\, ds$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$.

Example 6: Evaluate $\int_C 4y\, ds$, where C consists of the line segment from $(-2,-2)$ to $(0,0)$, followed by the arc of the parabola $x = y^2$ from $(0,0)$ to $(4,2)$.



$C = C_1 + C_2$

$C_1: \vec{r}_1(t) = \langle t, t\rangle, \quad -2 \le t \le 0$

$C_2: y(t) = t, \quad x(t) = t^2$

$\vec{r}_2 = \langle t^2, t\rangle, \quad 0 \le t \le 2$

$\int_C 4y\, ds = I_1 + I_2$, where $I_1 = \int_{C_1} 4y\, ds$ and $I_2 = \int_{C_2} 4y\, ds$

$\|\vec{r}_1\,'(t)\| = \|\langle 1,1\rangle\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\|\vec{r}_2\,'(t)\| = \|\langle 2t, 1\rangle\| = \sqrt{(2t)^2 + 1^2} = \sqrt{4t^2+1}$

$$I_1 = \int_{-2}^{0} 4t\|\vec{r}_1\,'(t)\|\, dt = \int_{-2}^{0} 4t\sqrt{2}\, dt$$

$$I_2 = \int_0^2 4t\|\vec{r}_2\,'(t)\|\, dt = \int_0^2 4t\sqrt{4t^2+1}\, dt$$

Line integrals of vector fields: $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\,ds$

Definition: Let **F** be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. The line integral of **F** on C is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big)\cdot\mathbf{r}'(t)\,dt .$$

$$\int_C \vec{F}\cdot\frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}}\,\cancel{\|\vec{r}\,'(t)\|}\,dt = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

An important application of this is *work* (the work done by a force in moving an object from one location to another). The work done by a force field **F** in moving an object along path C is:

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Example 7: Find the work done by $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ in moving an object along the line from $(0,0,0)$ to $(5,3,2)$.

Parametrize the path: $\vec{r}(t) = \langle 0,0,0\rangle + t\langle 5,3,2\rangle = \langle 5t, 3t, 2t\rangle$, $t \in [0,1]$

$\vec{F}\,'(t) = \langle 5,3,2\rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{25+9+4} = \sqrt{38}$

$\vec{F}(x(t), y(t), z(t)) = \vec{F}(5t, 3t, 2t) = 3t(2t)\,\hat{\imath} + 5t(2t)\,\hat{\jmath} + 5t(3t)\,\hat{k}$

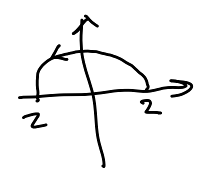
$= 6t^2\,\hat{\imath} + 10t^2\,\hat{\jmath} + 15t^2\,\hat{k}$

$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 6t^2, 10t^2, 15t^2\rangle\cdot\langle 5,3,2\rangle\,dt = \int_0^1 (30t^2 + 30t^2 + 30t^2)\,dt$

$= \int_0^1 90t^2\,dt = \frac{90t^3}{3}\Big|_0^1 = \frac{90}{3}$

$= \boxed{30}$

Example 8: Find the work done by $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$ in moving an object counterclockwise along the semicircle $y = \sqrt{4-x^2}$ from $(2,0)$ to $(-2,0)$.



$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le \pi$

$\vec{F} = \langle -y, -x\rangle = \langle -2\sin t, -2\cos t\rangle$

$d\vec{r} = \vec{r}\,'(t)\,dt = \langle -2\sin t, 2\cos t\rangle\,dt$

$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^\pi \langle -2\sin t, -2\cos t\rangle\cdot\langle -2\sin t, 2\cos t\rangle\,dt$

$= \int_0^\pi (4\sin^2 t - 4\cos^2 t)\,dt = -4\int_0^\pi (\cos^2 t - \sin^2 t)\,dt$

$= -4\int_0^\pi \cos(2t)\,dt = -4\left(\frac{1}{2}\right)\sin 2t\Big|_0^\pi = -2(\sin 2\pi - \sin 0)$

$= -2(0-0) = \boxed{0}$

Line integrals in differential form:

Suppose that $\mathbf{F}(x,y) = M\,\mathbf{i} + N\,\mathbf{j} + P\mathbf{k}$ and that $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (M\,dx + N\,dy + P\,dz).$$

Example 9: Evaluate $\int_C (3y - x)\,dx + y^2\,dy$, where C is the path given by $x = 2t$, $y = 10t$, $0 \le t \le 1$.

$x = 2t$ $\quad y = 10t$

$\frac{dx}{dt} = 2$ $\quad \frac{dy}{dt} = 10$

$dx = 2\,dt$ $\quad dy = 10\,dt$

$$\int_C (3y - x)\,dx + y^2\,dy$$

$$\int_0^1 (3(10t) - 2t)\,2dt + \int_0^1 (10t)^2\,10\,dt$$

$$\int_0^1 (56t)\,dt + \int_0^1 1000\,t^2\,dt = 56 \cdot \frac{t^2}{2}\Big|_0^1 + 1000 \cdot \frac{t^3}{3}\Big|_0^1$$

$$= \frac{1084}{3}$$

Example 10: Evaluate $\int_C xy\,dx + (x+y)\,dy$ along the curve $y = x^2$ from $(-1,1)$ to $(2,4)$.