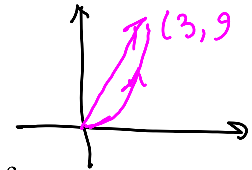


15.3: Conservative Vector Fields and Independence of Path

Example 1: Find the work done by $\mathbf{F}(x,y)=15x^2y^2\,\mathbf{i}+10x^3y\,\mathbf{j}$ in moving a particle from $(0,0)$ to $(3,9)$ along the following two paths:

Recall:

(a) $\mathbf{r}_1(t)=\langle t,3t\rangle,\ 0\le t\le 3$;

(b) $\mathbf{r}_2(t)=\langle t,t^2\rangle,\ 0\le t\le 3$.

Work $= \int_C \vec{F}\cdot d\vec{r}$, where $d\vec{r}=r'(t)\,dt$

$\vec{F}(x,y) = \langle 15x^2y^2, 10x^3y\rangle$

(a) $\vec{r}_1'(t) = \langle 1,3\rangle$

$d\vec{r}_1(t) = \langle 1,3\rangle\,dt$

Work $= \int_0^3 \vec{F}\cdot d\vec{r}_1 = \int_0^3 \langle 15(t)^2(3t)^2, 10(t)^3(3t)\rangle\cdot\langle 1,3\rangle\,dt$

$= \int_0^3 \langle 135t^4, 30t^4\rangle\cdot\langle 1,3\rangle\,dt = \int_0^3 (135t^4+90t^4)\,dt$

$= \int_0^3 (225t^4)\,dt = 225\cdot\frac{t^5}{5}\Big|_0^3 = 45(3^5-0) = 45(243) = \boxed{10935}$

(b) $\vec{r}_2(t) = \langle t,t^2\rangle \Rightarrow r_2'(t) = \langle 1,2t\rangle$

Work $= \int_0^3 \langle 15t^2(t^2)^2, 10t^3(t^2)\rangle\cdot\langle 1,2t\rangle\,dt = \int_0^3 \langle 15t^6, 10t^5\rangle\cdot\langle 1,2t\rangle\,dt$

$= \int_0^3 (15t^6+20t^6)\,dt = \int_0^3 35t^6\,dt = \frac{35t^7}{7}\Big|_0^3 = 5(3^7-0)$

$= \boxed{10935}$

<u>Theorem</u>: <u>Fundamental Theorem of Line Integrals</u>

Suppose R is an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$ that contains the piecewise smooth curve C, given by $\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}$ or $\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$, $a\le t\le b$.

Suppose that $\mathbf{F}(x,y)=M\,\mathbf{i}+N\,\mathbf{j}$ is conservative and that the component functions M and N are continuous. (Or, in $\mathbb{R}^3$, that $\mathbf{F}(x,y,z)=M\,\mathbf{i}+N\,\mathbf{j}+P\,\mathbf{k}$ is conservative and M, N, and P are continuous.) Then,

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}=f\big(x(b),y(b)\big)-f\big(x(a),y(a)\big)\quad (\text{in } \mathbb{R}^2),$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla f\cdot d\mathbf{r}=f\big(x(b),y(b),z(b)\big)-f\big(x(a),y(a),z(a)\big)\quad (\text{in } \mathbb{R}^3)$$

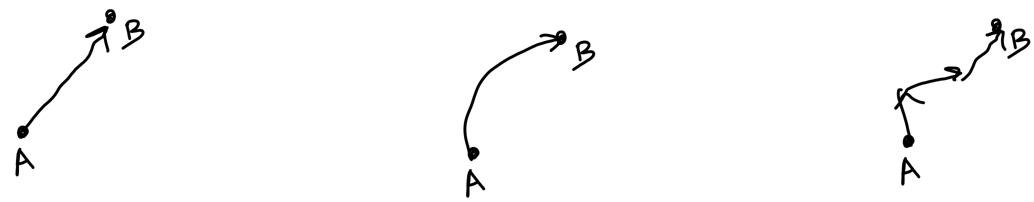
where f is a potential function of $\mathbf{F}$ (i.e., $\mathbf{F}(x,y)=\nabla f(x,y)$ or $\mathbf{F}(x,y,z)=\nabla f(x,y,z)$).

Yes!

Is the vector field in Example 1 conservative?

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(15x^2y^2) = 30x^2y$ $\qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(10x^3y) = 30x^2y$

Note: This means that the work done by a conservative vector field is *independent of path*. No matter what path is used to move a particle from Point A to Point B, the work is the same.



Example 2: Apply the Fundamental Theorem of Line Integrals to Example 1.

$\mathbf{F}(x,y) = 15x^2y^2\,\mathbf{i} + 10x^3y\,\mathbf{j}$ along paths (a) $\mathbf{r}_1(t) = \langle t, 3t\rangle$, $0 \le t \le 3$; (b) $\mathbf{r}_2(t) = \langle t, t^2\rangle$, $0 \le t \le 3$.

We already verified that $\vec{F}$ is conservative. Now find f such that $\nabla f = \vec{F}$.

$f(x,y) = \int f_x\,dx = \int M\,dx = \int 15x^2y^2\,dx = 15\frac{x^3}{3}y^2 + g(y) = 5x^3y^2 + g(y)$

$f(x,y) = \int f_y\,dy = \int N\,dy = \int 10x^3y\,dy = 10x^3\frac{y^2}{2} + h(x) = 5x^3y^2 + h(x)$

So, the potential function is $f(x,y) = 5x^3y^2 + K$

Work $= \int_{C_1} \vec{F}\cdot d\vec{r}_1 = \int_{C_2} \vec{F}\cdot d\vec{r}_2 = 5x^3y^2 \Big|_{(0,0)}^{(3,9)} = 5(3)^3(9)^2 - 5(0)^3(0)^2 = \boxed{10935}$

Example 3: Evaluate $\int_C (6x\,dx - 4z\,dy - (4y-20)\,dz)$ if C is a smooth curve from $(0,0,0)$ to $(3,4,0)$.

Equivalent problem: Find $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle 6x, -4z, -(4y-20)\rangle$, where C is a smooth curve from $(0,0,0)$ to $(3,4,0)$.

Also equivalent: Find the work done by $\vec{F} = \langle 6x, -4z, -(4y-20)\rangle$ on a particle moving from $(0,0,0)$ to $(3,4,0)$.

Is it conservative? If so, then $\text{curl}\,\vec{F} = \vec{0}$.

$$\text{curl}\,\vec{F}(x,y,z) = \nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6x & -4z & 20-4y \end{vmatrix} = \langle 0,0,0\rangle.$$

So yes, $\vec{F}$ is conservative. Find a potential function.

$f(x,y,z) = \int f_x\,dx = \int 6x\,dx = \frac{6x^2}{2} = 3x^2 + g(y,z)$

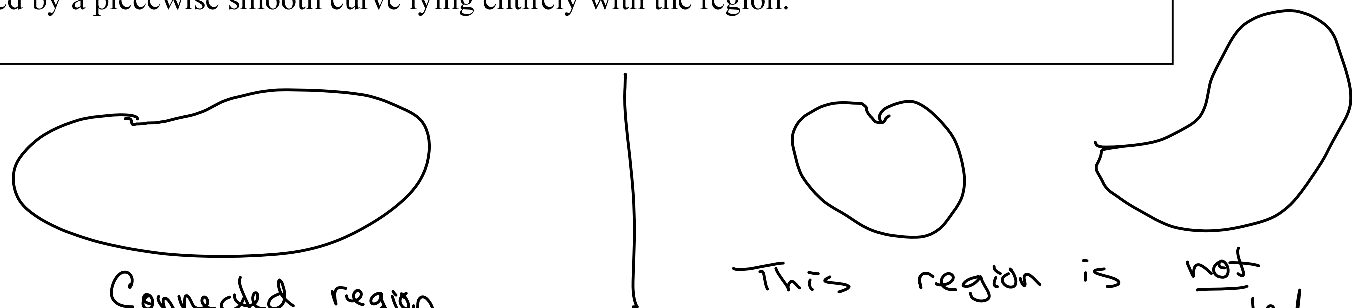
$f(x,y,z) = \int f_y\,dy = \int -4z\,dy = -4zy + h(x,z)$

$f(x,y,z) = \int f_z\,dz = \int (20-4y)\,dz = 20z - 4yz + l(x,y)$

So, $f(x,y,z) = 3x^2 - 4yz + 20z$. Now, apply the Fun. Thm of Line Integrals:

$\int_C \vec{F}\cdot d\vec{r} = f(3,4,0) - f(0,0,0) = 3(3)^2 - 4(4)(0) + 20(0) - 0 = \boxed{27}$

<u>Definition</u>: A region in $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be *connected* if any two points in the region can be joined by a piecewise smooth curve lying entirely with the region.



Connected region

This region is not connected.

If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is the same for every piecewise smooth curve from Point A to Point B, then we say that the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent of path*.

For open regions that are connected, the path independence of $\int_C \mathbf{F}\cdot d\mathbf{r}$ is equivalent to $\mathbf{F}$ being conservative.

<u>Theorem</u>:

If $\mathbf{F}$ is continuous on an open connected region in $\mathbb{R}^2$ or $\mathbb{R}^3$, then the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

This theorem, when combined with the Fundamental Theorem of Line Integrals, results in the following:

<u>Theorem</u>:

Suppose $\mathbf{F}(x,y,z) = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ has continuous first partial derivatives in an open connected region *R*, and suppose that *C* is any piecewise smooth curve in *R*. Then the following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function *f*.
2. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve *C* in *R*.

if $\vec{F} = \nabla f$, then $\int_C \vec{F}\cdot d\vec{r} = \underbrace{f(a_2,b_2,c_2) - f(a_1,b_1,c_1)}_{\text{This will be } 0}$

if $(a_2,b_2,c_2) = (a_1,b_1,c_1)$.

Note: A *closed curve* is a curve in which the beginning and ending points are the same. That is, if the curve C is given by $\mathbf{r}(t),\ a \le t \le b$, then $\mathbf{r}(a) = \mathbf{r}(b)$.

a

Example 4: Evaluate $\int_C (\sin y\,dx + x\cos y\,dy)$ if C is a smooth curve from $\left(3, \frac{\pi}{2}\right)$ to $\left(-7, \frac{\pi}{4}\right)$.

b $\left(126, \frac{79\pi}{13}\right)$ to $\left(126, \frac{79\pi}{13}\right)$

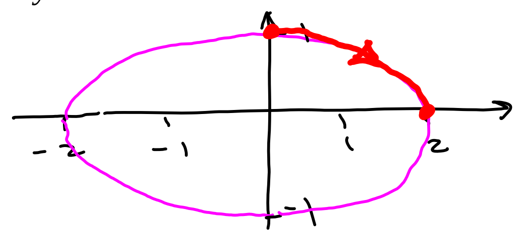
Is $\vec{F}$ conservative?

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin y) = \cos y \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\cos y) = \cos y \end{aligned}\right\} \vec{F} \text{ is conservative}$$

a see summer notes

b $\int_C \vec{F}\cdot d\vec{r} = 0$, because $\vec{F}$ is conservative and the beginning and ending points are the same.

Example 5: Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle 8xy - 12x^3, 4x^2 - 4y\rangle$ and C is the path from $(0,1)$ to $(2,0)$, along the first-quadrant portion of the ellipse $\frac{x^2}{4} + y^2 = 1$.



Method 1: Parametrize C.

$x(t) = 2\sin t$
$y(t) = \cos t$

$\vec{r}(t) = \langle 2\sin t, \cos t\rangle, \quad 0 \le t \le \frac{\pi}{2}$

$\vec{r}(0) = \langle 0, 1\rangle, \quad \vec{r}\left(\frac{\pi}{2}\right) = \langle 2, 0\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \vec{F}\cdot \vec{r}\,'(t)\,dt = \int_0^{\pi/2} \langle 8(2\sin t)\cos t - 12(2\sin t)^3, 4(2\sin t)^2 - 4\cos t\rangle \cdot \langle 2\cos t, -\sin t\rangle\,dt$$

$$= \int_0^{\pi/2} \langle 16\sin t\cos t - 96\sin^3 t, 16\sin^2 t - 4\cos t\rangle \cdot \langle 2\cos t, -\sin t\rangle\,dt$$

$$= \int_0^{\pi/2} (32\sin t\cos^2 t - 192\sin^3 t\cos t - 16\sin^3 t + 4\cos t\sin t)\,dt$$

Yuk! If $\vec{F}$ is conservative, we have other options.

Is it conservative $M = 8xy - 12x^3$, $N = 4x^2 - 4y$

$\frac{\partial M}{\partial y} = 8x, \quad \frac{\partial N}{\partial x} = 8x.$ Yes! It is conservative.

So, we have at least 2 other options:

Method 2: Find a potential function and apply the Fundamental Theorem of Line Integrals.

Method 3: Choose an easier path (probably a line). (Because $\vec{F}$ is conservative, $\vec{F}\cdot d\vec{r}$ is independent of path.)

Both of these are worked out in Summer Notes.