15.4: Green's Theorem

Definition: A curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j},\ a \le t \le b$, is said to be *simple* if $\mathbf{r}(c) \ne \mathbf{r}(d)$ for every c, d in $[a,b]$.

$\vec{r}(c) \neq \vec{r}(d)$

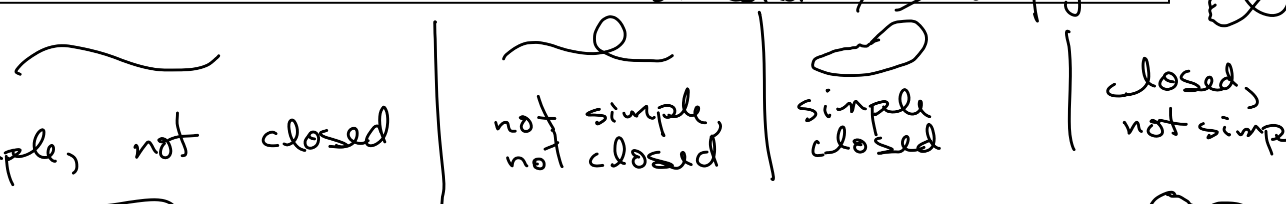
That is, a *simple curve* is a curve that does not intersect itself between its endpoints.

A connected region in $\mathbb{R}^2$ is said to be *simply connected* if every closed curve in R encloses only points that are in R.

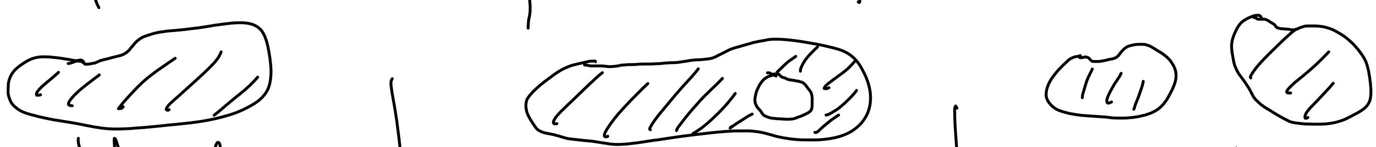
That is, a *simply connected region* does not have holes. (Also, it must be connected—it cannot consist of multiple disjoint pieces.)

simply connected $\Rightarrow$ connected

connected $\not\Rightarrow$ simply connected



simple, not closed | not simple, not closed | simple closed | closed, not simple



connected and simply connected | connected but not simply connected | not connected

Green's Theorem:

Suppose R is a simply connected region in $\mathbb{R}^2$ with a piecewise smooth boundary C, oriented counterclockwise.

counterclockwise $\Longleftrightarrow$ positively oriented

(That is, the region R lies to the left as C is traversed exactly once.)

Suppose $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ is a vector field with M and N having continuous first partial derivatives in an open region containing R. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

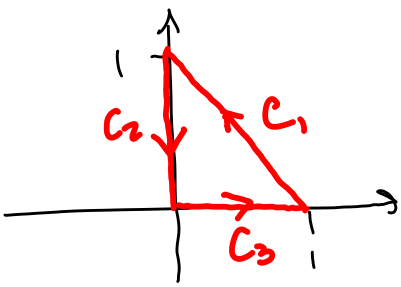
Note on Notation: An integral with a circle indicates that the line integral is evaluated over a simple closed curve. Sometimes an arrow is used to indicate the orientation.

$$\oint_C M\,dx + N\,dy \quad \text{or} \quad \oint_C M\,dx + N\,dy$$

$$\oint_C M\,dx + N\,dy \quad \text{or} \quad \oint M\,dx + N\,dy$$

Example 1: Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, traversed counterclockwise.

Work this problem 2 ways: ① evaluating the line integral directly
② Applying Green's Thm



Is $\vec{F}$ conservative?

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^4) = 0$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(xy) = y$$

Nope, not conservative.

Method: Evaluate line integral directly.

Parametrize C.

C_1: $\vec{r}_1(t) = \langle 1-t, t\rangle$, $0 \le t \le 1$

C_2: $\vec{r}_2(t) = \langle 0, 2-t\rangle$, $1 \le t \le 2$

C_3: $\vec{r}_3 = \langle t-2, 0\rangle$, $2 \le t \le 3$.

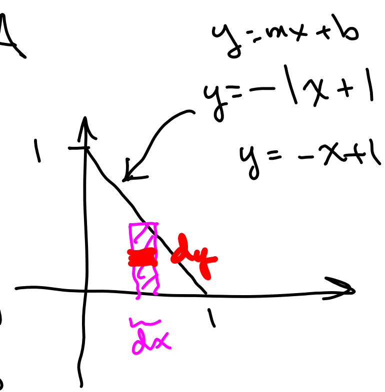
Then evaluate 3 different line integrals and add them.

$I_1 = \int_0^1 \vec{F}(\vec{r}_1(t))\cdot d\vec{r}_1$, $I_2 = \int_1^2 \vec{F}(\vec{r}_2(t))\cdot d\vec{r}_2$, $I_3 = \int_2^3 \vec{F}(\vec{r}_3(t))\cdot d\vec{r}_3$

See summer notes for details.

method 2: Green's Theorem

$$\int_C x^4\,dx + xy\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$



$$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2}\Big|_0^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{2}\cdot\frac{(1-x)^3}{3}\Big|_0^1$$

$$= -\frac{1}{6}(0^3 - 1^3) = \boxed{\frac{1}{6}}$$

Example 2: Evaluate $\int_C (y - x^2)\,dx + (2x - y^2)\,dy$, where C is the boundary of the region lying inside the semicircle $y = \sqrt{25 - x^2}$ and outside the semicircle $y = \sqrt{9 - x^2}$, traversed counterclockwise (positively oriented).

Example 3: Evaluate $\int_C (y + e^{\sqrt{x}})\,dx + (x^2 + \cos y^2)\,dy$, where C is the positively oriented boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Example 4: Evaluate the work done by the force field $\mathbf{F}(x, y) = xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ on a particle traversing (in a counterclockwise direction) the boundary of the square with vertices $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

Using Green's Theorem to find area:

Sometimes the double integral over an area is easier to calculate than the line integral around the boundary. Other times, the reverse is true.

We can choose *M* and *N* strategically to come up with a formula for area:

To find area, we want $\int_R \int 1 \, dA$ Want to find M and N such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$. Lots of ways to do it,

One way:

$\frac{\partial M}{\partial y} = -\frac{1}{2}$, $\frac{\partial N}{\partial x} = \frac{1}{2}$

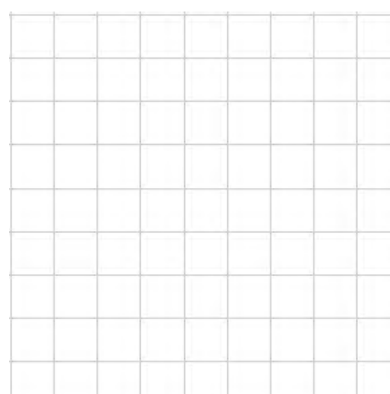
$M = -\frac{1}{2}y$, $N = \frac{1}{2}x$

$\int_C -\frac{1}{2}y \, dx + \frac{1}{2}x \, dy = \frac{1}{2}\int x\,dy - y\,dx$

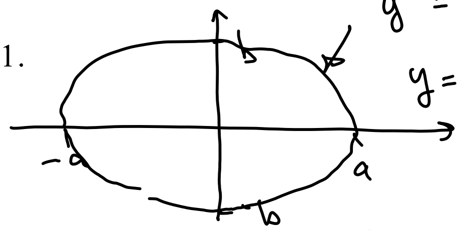
> Theorem: Line Integral for Area
>
> Suppose *R* is a simply connected region in $\mathbb{R}^2$ with a piecewise smooth boundary *C*, oriented counterclockwise. Then the area of *R* is
>
> $$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx.$$

Example 5: Find the area of the triangle with vertices $(1,2)$, $(7,3)$, and $(6,1)$.

A lot more hassle to calculate the line integral around the triangle than to calculate the area directly.



Example 6: Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$

$y = \sqrt{b^2(1-x^2)}$

Let $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$

$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx$

$= \frac{1}{2}\int_0^{2\pi} a\cos t\,(b\cos t\,dt) - b\sin t\,(-a\sin t)\,dt$

$= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt$

$= \frac{1}{2}ab\int_0^{2\pi} 1\,dt$

$\frac{dx}{dt} = -a\sin t$

$dx = -a\sin t\,dt$

$\frac{dy}{dt} = b\cos t\,dt$

$dy = b\cos t\,dt$

$\frac{1}{2}ab\,t\Big|_0^{2\pi}$

$= \frac{1}{2}ab(2\pi - 0)$

$= \boxed{\pi ab}$

Extending Green's Theorem to a region with a hole:

Example 7: Evaluate $\int_C (y-3x^2)\,dx + (2x-\sin y)\,dy$, where C is the boundary of the region lying between the circles $x^2+y^2=4$ and $x^2+y^2=1$.