11.7: Cylindrical and Spherical Coordinates

Cylindrical coordinates:

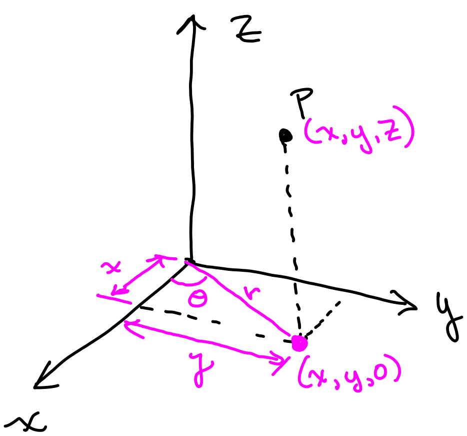
Cylindrical coordinates extend the polar coordinate system into $\mathbb{R}^3$.

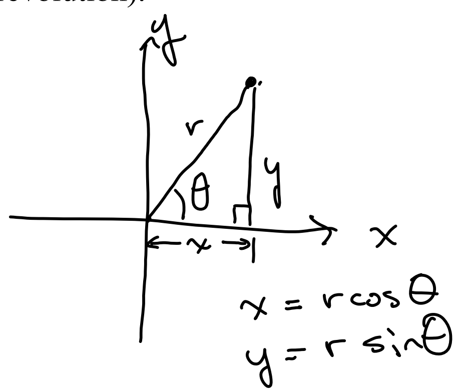
$$\underset{\text{Rectangular}}{P(x,y,z)} \longrightarrow \underset{\text{Cylindrical}}{P'(r,\theta,z)}$$

In a cylindrical coordinate system, a point P in $\mathbb{R}^3$ is represented by an ordered triple (r,θ,z) .

1. (r,θ) is a polar representation of the projection of P in the xy-plane.
2. z has the same meaning as in rectangular coordinates.

Note: Cylindrical coordinates are especially useful for representing surfaces for which the z-axis is the axis of symmetry (cylindrical surfaces and surfaces of revolution).





Converting between rectangular and cylindrical coordinate systems:		
$x = r\cos\theta$	$y = r\sin\theta$	$z = z$
$x^2 + y^2 = r^2$	$\tan\theta = \dfrac{y}{x}$	$z = z$

Example 1: Convert the point with rectangular coordinates $(-2\sqrt{2}, 2\sqrt{2}, 2)$ to cylindrical coordinates.

$x = -2\sqrt{2}$, $y = 2\sqrt{2}$, $z = 2$

$\tan\theta = \frac{y}{x} = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1 \Rightarrow \theta = \frac{3\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4}, \ldots$

$x^2+y^2 = r^2 \Rightarrow r^2 = (-2\sqrt{2})^2 + (2\sqrt{2})^2$
$= 4\cdot 2 + 4\cdot 2$
$= 16.$ So $r = \pm 4$

Must be in 2nd quadrant ($x<0$, $y>0$), so choose $r = 4$, $\theta = \frac{3\pi}{4}$

$$\boxed{(r,\theta,z) = \left(4, \frac{3\pi}{4}, 2\right)}$$

Example 2: Convert the point with cylindrical coordinates $\left(3, \frac{\pi}{3}, -5\right)$ to rectangular coordinates.

$r = 3, \ \theta = \frac{\pi}{3}$

$x = r\cos\theta = 3\cos\frac{\pi}{3} = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$y = r\sin\theta = 3\sin\frac{\pi}{3} = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

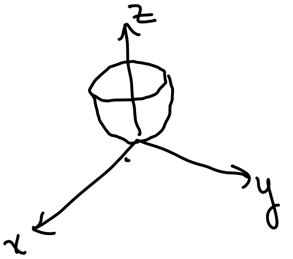
$\boxed{(x, y, z) = \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, -5\right)}$

Example 3: Convert the equation $z = x^2 + y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Traces:

$z = 0 \Rightarrow 0 = x^2 + y^2$ Pt $(0,0)$

$\left.\begin{array}{l} x = 0 \Rightarrow z = y^2 \\ y = 0 \Rightarrow z = x^2 \end{array}\right\}$ parabolas



circular paraboloid

$x^2 + y^2 = r^2$

$\Downarrow$

$\boxed{z = r^2}$

$z = k \Rightarrow k = r^2$

$(k > 0)$ $k = x^2 + y^2$ circle

Example 4: Convert the equation $z = x^2 - y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

$z = x^2 - y^2$

$x = r\cos\theta, \ y = r\sin\theta \Rightarrow z = (r\cos\theta)^2 - (r\sin\theta)^2$

$z = r^2\cos^2\theta - r^2\sin^2\theta$

$z = r^2(\cos^2\theta - \sin^2\theta)$

Using double-angle identity: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

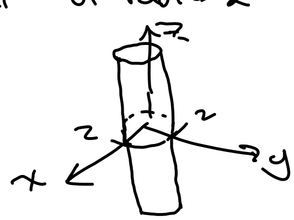
$\boxed{z = r^2\cos(2\theta)}$

Example 5: Convert the equation $4 = x^2 + y^2$ in rectangular coordinates into an equation in cylindrical coordinates.

Cylinder of radius 2

$4 = r^2$

$\boxed{r = 2}$ $(4 = r^2 \Rightarrow r = \pm 2$, but we choose $r = 2)$

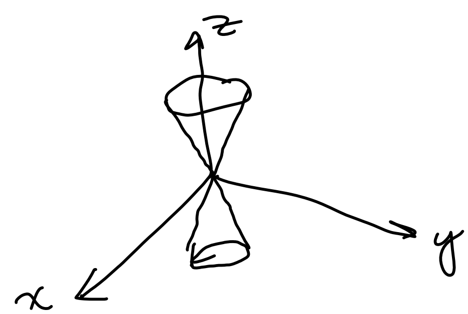


Example 6: Convert the equation $x^2 + y^2 = z^2$ in rectangular coordinates into an equation in cylindrical coordinates.

$r^2 = z^2$

$z = \pm r$

Circular cone



Example 7: Convert the equation $r = 2\cos\theta$ in cylindrical coordinates into an equation in rectangular coordinates.

$r = 2\cos\theta$

Multiply by r: $r^2 = 2r\cos\theta$

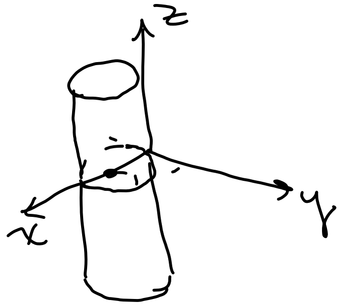
$$\boxed{x^2+y^2 = 2x}$$

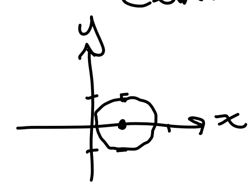
To sketch it, complete the square: $x^2 - 2x + y^2 = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Circle of radius 1, center (1,0)





Spherical coordinates:

$P(x,y,z) \longrightarrow P'(\rho,\theta,\phi)$

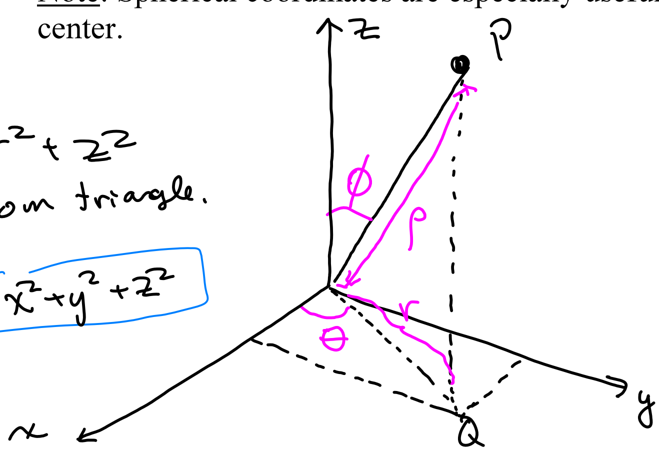
Rectangular → Spherical

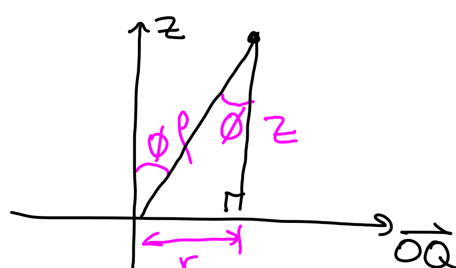
In a spherical coordinate system, a point P in $\mathbb{R}^3$ is represented by an ordered triple (ρ,θ,ϕ).

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the angle between the positive x-axis and the projection of $\overrightarrow{OP}$ in the xy-plane (same θ as in cylindrical coordinates).
3. ϕ is the angle between the positive z-axis and $\overrightarrow{OP}$ $(0 \leq \phi \leq \pi)$

↳ phi ϕ

Note: Spherical coordinates are especially useful for surfaces that are symmetric about a point, or center.





$\rho^2 = r^2 + z^2$ from triangle.

so, $\boxed{\rho^2 = x^2+y^2+z^2}$

$\frac{r}{\rho} = \sin\phi$

$r = \rho\sin\phi$

$\frac{z}{\rho} = \cos\phi$

$z = \rho\cos\phi$

$\boxed{\cos\phi = \frac{z}{\rho}}$

$\rho = \sqrt{x^2+y^2+z^2}$

$\boxed{\cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}}$

$\boxed{x = \underbrace{\rho\sin\phi}_{r}\cos\theta}$

We know $x = r\cos\theta$

From $y = r\sin\theta$ substitute $r = \rho\sin\phi$ gives us $\boxed{y = \underbrace{\rho\sin\phi}_{r}\sin\theta}$

<u>Converting between rectangular and spherical coordinate systems</u>:

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

$$x^2 + y^2 + z^2 = \rho^2 \qquad \tan\theta = \frac{y}{x} \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

<u>Converting between cylindrical and spherical coordinate systems</u> $r \geq 0$:

$$r^2 = \rho^2 \sin^2\phi \qquad \theta = \theta \qquad z = \rho\cos\phi$$

$\hookrightarrow r = \rho \sin\phi$

$$\rho = \sqrt{r^2 + z^2} \qquad \theta = \theta \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{r^2+z^2}}\right)$$

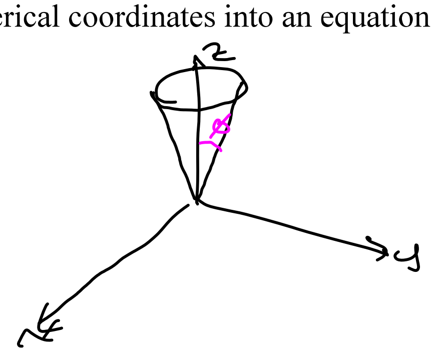
Example 8: Convert the equation $\rho = c$ (c a constant) in spherical coordinates into an equation in rectangular coordinates.

$$\rho = c$$
$$\rho^2 = c^2$$
$$x^2 + y^2 + z^2 = c^2$$

(A sphere of radius c)

Example 9: Convert the equation $\phi = c$, $0 < c < \frac{\pi}{2}$, in spherical coordinates into an equation in rectangular coordinates.

$$\cos\phi = \frac{z}{\rho}$$
$$\cos c = \frac{z}{\rho}$$
$$\cos c = \frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$(x^2+y^2+z^2)\cos^2 c = z^2$$
$$x^2+y^2+z^2 = \frac{z^2}{\cos^2 c}$$



$$\to x^2 + y^2 = \frac{z^2}{\cos^2 c} - z^2$$
$$x^2 + y^2 = z^2 \underbrace{\left(\frac{1}{\cos^2 c} - 1\right)}_{\text{constant}}$$

Example 10: Convert the equation $x^2 + y^2 - 3z^2 = 0$ to spherical coordinates from rectangular coordinates.

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$ $\qquad$ $z = \rho\cos\phi$

$x = \underbrace{\rho\sin\phi}_{r}\cos\theta$ $\qquad$ $y = \underbrace{\rho\sin\phi}_{r}\sin\theta$

$x^2 + y^2 - 3z^2 = 0$

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta - 3\rho^2\cos^2\phi = 0$

$\rho^2\sin^2\phi\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} - 3\rho^2\cos^2\phi = 0$

$\Rightarrow \rho^2\sin^2\phi - 3\rho^2\cos^2\phi = 0$

$\rho^2\sin^2\phi = 3\rho^2\cos^2\phi$

either $\rho = 0$, or

$\sin^2\phi = 3\cos^2\phi$

$\frac{\sin^2\phi}{\cos^2\phi} = 3$

$\tan^2\phi = 3$

$\tan\phi = \pm\sqrt{3}$

$\tan\phi = \sqrt{3}$

$\Rightarrow \phi = \frac{\pi}{3}$

$\tan\phi = -\sqrt{3}$

$\Rightarrow \phi = \frac{2\pi}{3}$

$\boxed{\phi = \frac{\pi}{3} \text{ or } \phi = \frac{2\pi}{3}}$

cone

Example 11: Describe the surface with equation $\phi = \dfrac{\pi}{2}$ in spherical coordinates.

See Summer 2015 Notes

Example 12: Describe the surface with equation $\phi = \dfrac{\pi}{4}$ in spherical coordinates.

Example 13: Convert the point that is represented by $(1, 2, 3)$ in rectangular coordinates to cylindrical and spherical coordinates.

Homework Qs

11.6 # 21) $x^2 - y^2 + z = 0 \Rightarrow z = y^2 - x^2$

Traces:

xy-plane: $z = 0$

$x^2 - y^2 = 0$

$x^2 = y^2 \Rightarrow y = \pm x$ lines

$z = k$: $k = y^2 - x^2$

$\frac{k}{k} = \frac{y^2}{k} - \frac{x^2}{k}$

$1 = \frac{y^2}{k} - \frac{x^2}{k}$ hyperbola

yz-plane: $x = 0$

$z = y^2$

parabola

$x = k \Rightarrow z = y^2 - k^2$

parabola

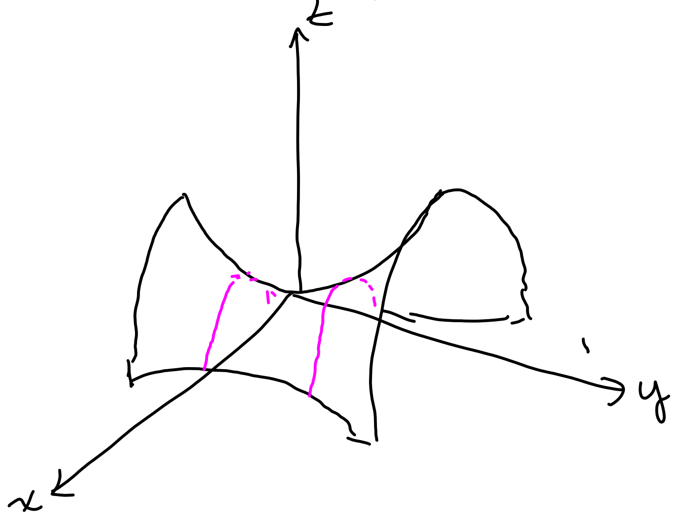
xz-plane: $y = 0$

$z = -x^2$ parabola

$y = k \Rightarrow z = k^2 - x^2$

parabola

hyperbolic paraboloid



$z = y^2 - x^2$