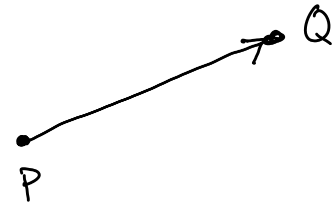
11.1: Vectors in the Plane

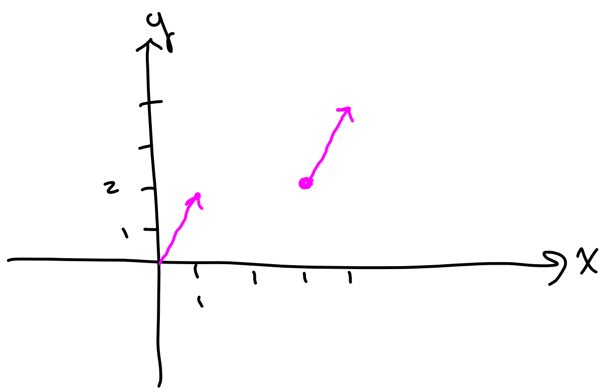
Note Title 8/25/2015

Vector: A directed line segment from an initial point to a terminal point.



P: initial point
Q: terminal point

Vectors are equal if they have the same length and the same direction.



These two vectors are equal.

Notation for vectors:

For type, they are in bold.

When handwritten, put arrows on top.

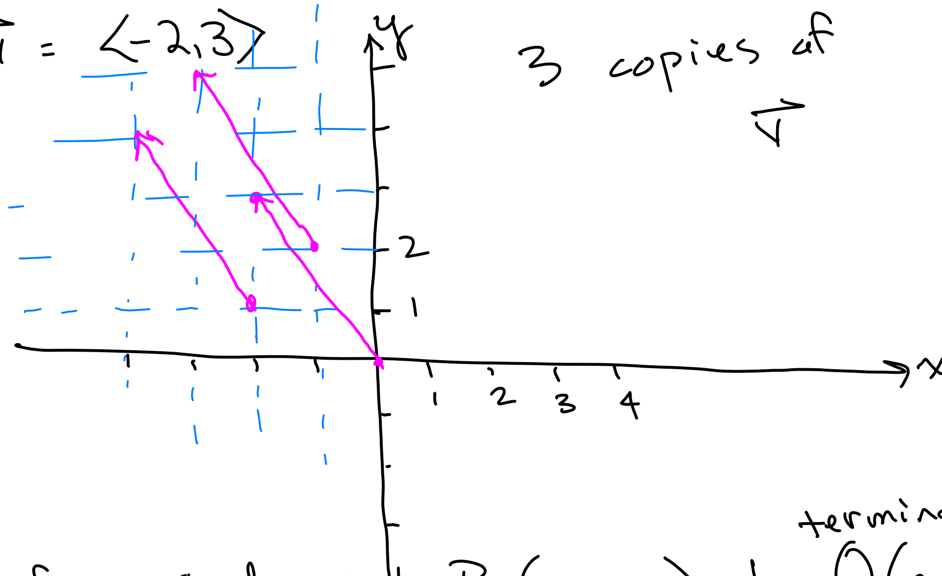
Example: $\overrightarrow{PQ}$ or $\vec{v}$ or $\vec{u}$

Component form for a vector:

The component form for a vector $\vec{v}$ is given by $\langle v_1, v_2 \rangle$, where v_1 is the horizontal distance from the initial point to the terminal point, and v_2 is the vertical distance.

<u>Note:</u> v_1 and v_2 are <u>scalars</u> (numbers), not vectors.

<u>Ex:</u> Suppose $\vec{v} = \langle -2, 3\rangle$



3 copies of $\vec{v}$

<u>Note:</u> The vector from initial point $P(p_1, p_2)$ to terminal pt $Q(q_1, q_2)$ is $\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2\rangle$.

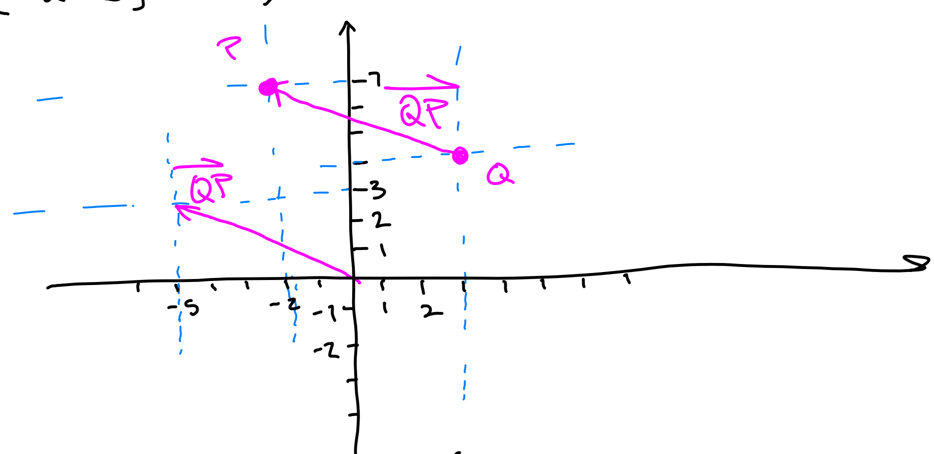
<u>Zero vector:</u> $\vec{0} = \langle 0, 0\rangle$

(length is 0; the terminal and initial points are the same)

<u>Example:</u> Suppose $P = (-2, 7)$ and $Q = (3, 4)$.

Find $\overrightarrow{QP}$.

$$\overrightarrow{QP} = \langle -2-3, 7-4\rangle = \langle -5, 3\rangle$$



Vector $\overrightarrow{QP}$ could have any initial point.

<u>Position vector for an object:</u> initial point is placed at the origin.

Magnitude (norm) of a vector: The length of the vector

For $\vec{v} = \langle v_1, v_2 \rangle$, the magnitude is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

Note: In one dimension (the real number line), the magnitude is the absolute value.

Ex: $|-7| = 7$

Unit vector: Any vector with magnitude 1.

Ex: Find the magnitude of $\vec{v} = \langle -5, 3 \rangle$. (as in example above)

$$\|\vec{v}\| = \|\langle -5, 3 \rangle\| = \sqrt{(-5)^2 + (3)^2}$$

$$= \sqrt{25 + 9} = \boxed{\sqrt{34}}$$

Note: magnitude is a scalar.

Vector operations

$\in$: "is an element of"

c is a real number

Definition

Suppose $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, $c \in \mathbb{R}$

$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$ (vector addition)

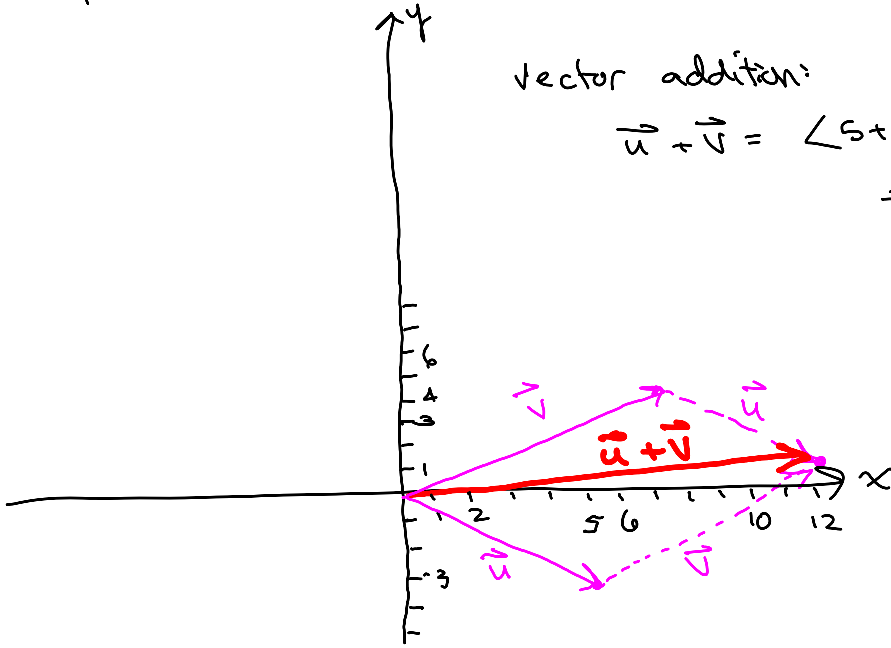
$c\vec{v} = \langle cv_1, cv_2 \rangle$ (scalar multiplication)

Combining these gives us vector subtraction:

$$\vec{u} - \vec{v} = \vec{u} + (-1\vec{v}) = \langle u_1, u_2 \rangle + \langle -v_1, -v_2 \rangle = \langle u_1 - v_1, u_2 - v_2 \rangle$$

Adding vectors geometrically: think of triangles and parallelograms.

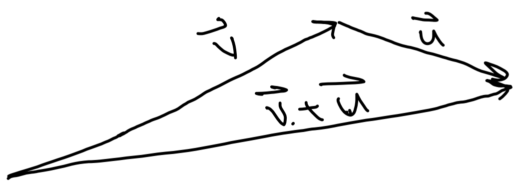
Example: Suppose $\vec{u} = \langle 5, -3 \rangle$ and $\vec{v} = \langle 7, 4 \rangle$.

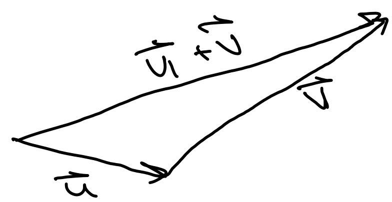


vector addition:

$\vec{u} + \vec{v} = \langle 5+7, -3+4 \rangle$

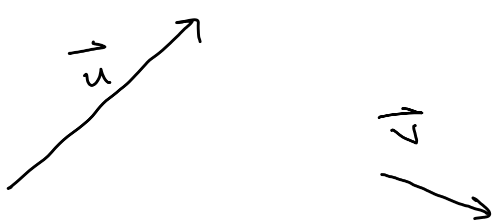
$= \langle 12, 1 \rangle$



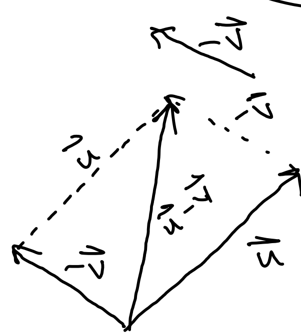


What about $\vec{u} - \vec{v}$?

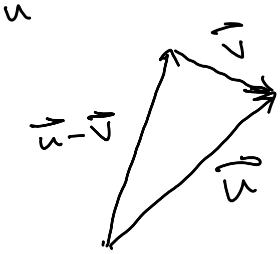
Given these graphs of $\vec{u}$ and $\vec{v}$, find $\vec{u} - \vec{v}$.



1st Draw $-\vec{v}$:



Put them together into a parallelogram.



Notice:

Note: cannot add scalars to vectors

Ex: $\langle -2,1\rangle + 3$ is nonsense.

Length of a scalar multiple.

Let $\vec{v}$ be a vecter and c a scalar.

Then, $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

Example: suppose $\vec{v} = \langle -2,5\rangle$ and $c=-3$.

$\|c\vec{v}\| = |-3|\,\|\langle -2,5\rangle\| = 3\sqrt{(-2)^2+5^2} = 3\sqrt{29}$

Also $\|c\vec{v}\| = \|-3\langle -2,5\rangle\| = \|\langle 6,-15\rangle\| = \sqrt{6^2+15^2} = \sqrt{36+225}$

$= \sqrt{261}$

$= \sqrt{9\cdot 29}$

$= \boxed{3\sqrt{29}}$

$$\begin{array}{r} 29 \\ 9\overline{)261} \\ \underline{18} \\ 81 \\ \underline{81} \\ 0 \end{array}$$

Unit vectors:

To normalize a vector means to find a unit vector in the same direction as the original vector.

☆ The unit vector in the direction of $\vec{v}$ is

$$\frac{\vec{v}}{\|\vec{v}\|}.$$

(magnitude of a unit vector is 1).

Example: Suppose $\vec{v} = \langle 2,-3\rangle$. Find a unit vector in

(a) the same direction as $\vec{v}$.

(b) the opposite direction as $\vec{v}$.

(a) First find the magnitude:

$$\|\vec{v}\| = \|\langle 2,-3\rangle\| = \sqrt{4+9} = \sqrt{13}$$

unit vector in same direction as $\vec{v}$

is $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle 2,-3\rangle}{\sqrt{13}} = \boxed{\left\langle \dfrac{2}{\sqrt{13}}, -\dfrac{3}{\sqrt{13}}\right\rangle}$

$\frac{1}{\sqrt{13}}\langle 2,-3\rangle$

Check that the magnitude is 1:

$$\|\vec{u}\| = \left\|\left\langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right\rangle\right\| = \sqrt{\left(\frac{2}{\sqrt{13}}\right)^2 + \left(\frac{-3}{\sqrt{13}}\right)^2} = \sqrt{\frac{4}{13}+\frac{9}{13}} = \sqrt{\frac{13}{13}}$$
$$= \sqrt{1}$$
$$= 1 \checkmark$$

(b) unit vector in the opposite direction of $\vec{v}$ is

$$\boxed{\vec{w} = \left\langle -\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right\rangle}$$

Ex: Find the vector $\vec{v}$ with magnitude 2, and in the same direction as $\vec{u} = \langle\sqrt{3}, 3\rangle$.

Find a unit vector in same direction as $\vec{u}$ (normalize $\vec{u}$)

$$\|\vec{u}\| = \sqrt{(\sqrt{3})^2 + 3^2} = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$$

unit vector: $\dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{\langle\sqrt{3},3\rangle}{2\sqrt{3}} = \left\langle \dfrac{\sqrt{3}}{2\sqrt{3}}, \dfrac{3}{2\sqrt{3}}\right\rangle = \left\langle \dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right\rangle$

Then multiply by $\|\vec{v}\| = 2$:

$$\vec{v} = 2\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \boxed{\langle 1, \sqrt{3}\rangle} \leftarrow \text{answer}$$

Check the magnitude is 2: $\|\vec{v}\| = \sqrt{1+3} = \sqrt{4} = 2 \checkmark$

Another approach:

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All unit vectors have length 1, so

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\frac{\langle \sqrt{3}, 3 \rangle}{2\sqrt{3}} = \frac{\vec{v}}{2}$$

multiply both sides by 2: $\frac{2\langle \sqrt{3}, 3 \rangle}{2\sqrt{3}} = \vec{v}$

$$\frac{\langle \sqrt{3}, 3 \rangle}{\sqrt{3}} = \vec{v}$$

$$\left\langle \frac{\sqrt{3}}{\sqrt{3}}, \frac{3}{\sqrt{3}} \right\rangle = \vec{v}$$

$$\boxed{\vec{v} = \langle 1, \sqrt{3} \rangle}$$

<u>Standard unit vectors:</u>

$\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$

sometimes written as $\hat{i} = \langle 1, 0 \rangle$, $\hat{j} = \langle 0, 1 \rangle$

<u>Example:</u> $\langle 5, -2 \rangle = 5\hat{i} - 2\hat{j}$

$$= 5\vec{i} - 2\vec{j}$$