11.1: Vectors (cont'd)

Note Title 8/27/2015

Recall: Every vector in $\mathbb{R}^2$ can be written 2 ways:

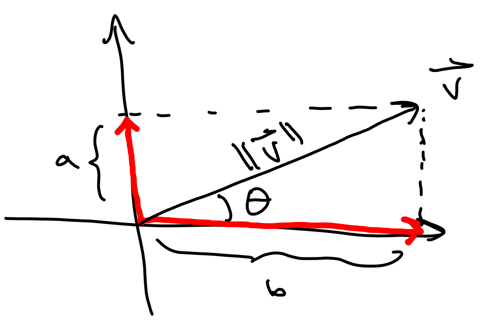
1) component form: $\vec{v} = \langle a, b \rangle$
2) As a linear combination of the standard unit vectors $\hat{\imath} = \vec{\imath} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \vec{\jmath} = \langle 0, 1 \rangle$.

Example: $\langle -3, -4 \rangle = -3\hat{\imath} - 4\hat{\jmath}$

Breaking a vector into horizontal and vertical components:

A vector $\vec{v}$ that makes a positive angle θ with the positive x-axis can be written as

$$\vec{v} = \langle \|\vec{v}\| \cos\theta, \|\vec{v}\| \sin\theta \rangle = \|\vec{v}\| \cos\theta \,\hat{\imath} + \|\vec{v}\| \sin\theta \,\hat{\jmath}$$



$$\cos\theta = \frac{b}{\|\vec{v}\|}$$

$$\|\vec{v}\| \cos\theta = b$$

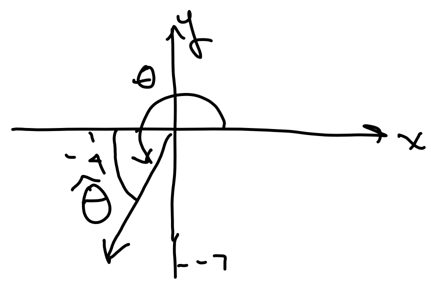
$$\sin\theta = \frac{a}{\|\vec{v}\|}$$

$$\|\vec{v}\| \sin\theta = a$$

Ex: Write the vector that has magnitude 5 and makes an angle of $115°$ with the positive x-axis.

$$\vec{v} = \langle 5\cos 115°, 5 \sin 115° \rangle \approx \boxed{\langle -2.11, 4.53 \rangle}$$

Ex: What angle does $\langle -4, -7 \rangle$ make with the positive x-axis?



$\hat{\theta}$ = reference angle

$$\|\vec{v}\| = \sqrt{4^2+7^2} = \sqrt{16+49} = \sqrt{65}$$

$$\cos\theta = \frac{-4}{\sqrt{65}}$$

$$\tan\hat{\theta} = \frac{7}{4} \Rightarrow \hat{\theta} = \tan^{-1}\left(\frac{7}{4}\right) \approx 60.26^\circ$$

So $\theta \approx 180^\circ + 60.26^\circ = \boxed{240.26^\circ}$

(approximate answer)

Exact answer:

$$\boxed{\theta = \pi + \tan^{-1}\left(\frac{7}{4}\right)}$$

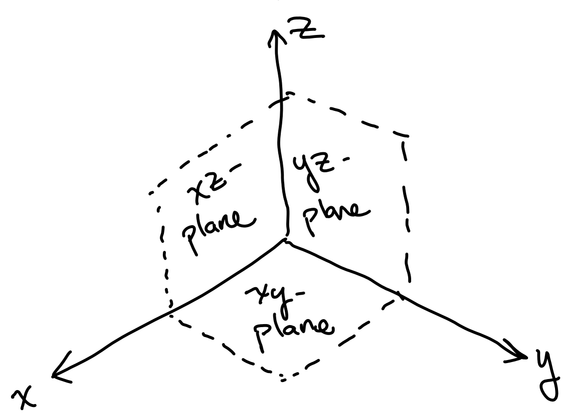
or $\theta = 180^\circ + \tan^{-1}\left(\frac{7}{4}\right)$, where $\tan^{-1}\left(\frac{7}{4}\right)$ is measured in degrees.

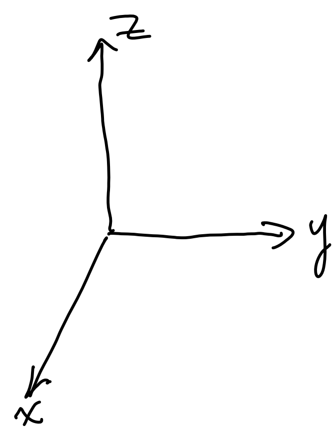
11.2: Coordinates and Vectors in $\mathbb{R}^3$

$\mathbb{R}^3$ = "space"

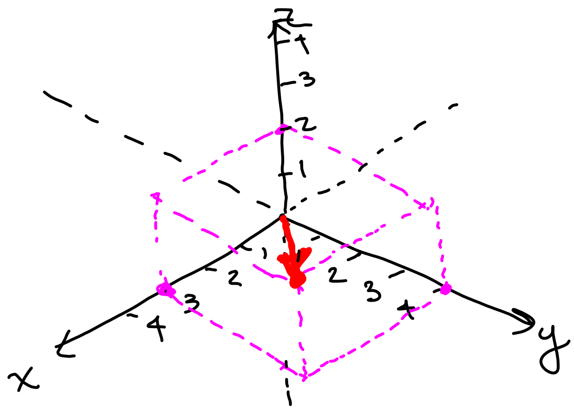
$\mathbb{R}$: set of all real numbers

3-dimensional coordinate system: Points are indicated by ordered triples (x, y, z)





Example: Plot the point $(3,4,2)$.



The coordinate planes divide $\mathbb{R}^3$ into 8 octants.

1st octant: all 3 coordinates are positive

Now draw the vector $\langle 3,4,2\rangle$.

Remember, this is the same vector, no matter where in $\mathbb{R}^3$ it is placed. Vectors have length and direction, but not location.

Position Vector: Vector from the origin to a point. (initial point at $(0,0,0)$)

In the picture, I've put $\langle 3,4,2\rangle$ as a position vector for the point $(3,4,2)$.

Distance Formula:

The distance between points (x_1,y_1,z_1) and (x_2,y_2,z_2) is:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Equation of a sphere:

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2,$$ where

r is the radius, and (x_1,y_1,z_1) is the center.

Midpoint Formula: The midpoint of the line segment joining (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$M=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

Example: Find the center and radius of the sphere that has equation

$$x^2+y^2+z^2+9x-2y+10z+19=0.$$

$$x^2+9x+\frac{81}{4} + y^2-2y+1 + z^2+10z+25 = -19+\frac{81}{4}+1+25$$

scratchwork

$\left(\frac{9}{2}\right)^2 = \frac{81}{4}$

$\left(\frac{-2}{2}\right)^2 = (-1)^2 = 1$

$\left(\frac{10}{2}\right)^2 = 5^2 = 25$

$7+\frac{81}{4} = \frac{28}{4}+\frac{81}{4}$

$= \frac{109}{4}$

$$\left(x+\frac{9}{2}\right)^2 + (y-1)^2 + (z+5)^2 = 7+\frac{81}{4}$$

$$\left(x+\frac{9}{2}\right)^2 + (y-1)^2 + (z+5)^2 = \frac{109}{4}$$ sphere in standard form.

$$\boxed{\begin{array}{l}\text{Radius: } \sqrt{\frac{109}{4}} = \frac{\sqrt{109}}{2} \\ \text{Center: } \left(-\frac{9}{2}, 1, -5\right)\end{array}}$$

Standard unit vectors in $\mathbb{R}^3$

$\vec{i} = \hat{i} = \langle 1,0,0\rangle$

$\vec{j} = \hat{j} = \langle 0,1,0\rangle$

$\vec{k} = \hat{k} = \langle 0,0,1\rangle$

If $\vec{v} = \langle v_1, v_2, v_3\rangle$, the unit vector in the same direction is $\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|}\langle v_1, v_2, v_3\rangle$, where $\|\vec{v}\| = \sqrt{v_1^2+v_2^2+v_3^2}$

Parallel vectors:

Definition: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if there is a scalar c such that $\vec{u} = c\vec{v}$.

Ex: Are these two vectors parallel?

$u = \langle 2,-3,4\rangle$ and $\vec{v} = \langle -6,9,-12\rangle$.

Yes, $-3\vec{u} = -3\langle 2,-3,4\rangle = \langle -6,9,-12\rangle = \vec{v}$.

Triangle Inequality (Applies to both $\mathbb{R}^2$ and $\mathbb{R}^3$)

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Ex: Are these points collinear?

Recall: Collinear: All can be connected with the same line.

A $(4,-2,7)$
B $(-2,0,3)$
C $(7,-3,9)$

$\vec{AB} = \langle -2-4,\ 0-(-2),\ 3-7\rangle = \langle -6, 2, -4\rangle$

$\vec{BC} = \langle 7-(-2),\ -3-0,\ 9-3\rangle = \langle 9,-3,6\rangle$

$\vec{AC} = \langle 7-4,\ -3-(-2),\ 9-7\rangle = \langle 3,-1,2\rangle$

Note that: $3\vec{AC} = \vec{BC}$. Also $-2\vec{AC} = \vec{AB}$. So $\vec{AC} \parallel \vec{BC}$ (is parallel to) and $\vec{AC}$ is parallel to $\vec{AB}$

So they must all be parallel.
So the points must be collinear.

$\|\vec{AB}\| = \sqrt{36+4+16} = \sqrt{56} = \sqrt{14\cdot 4} = 2\sqrt{14}$

$\|\vec{BC}\| = \sqrt{81+9+36} = \sqrt{126} = \sqrt{9\cdot 14} = 3\sqrt{14}$

$\|\vec{AC}\| = \sqrt{9+1+4} = \sqrt{14}$

Notice: $\|\vec{AC}\| + \|\vec{AB}\| = \|\vec{BC}\|$, so they must be collinear.

11.3: The Dot Product (of 2 vectors)

Definition of the Dot Product:

For $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, the dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2.$$

For $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example: $\langle 4, -2, 3 \rangle \cdot \langle 2, 5, -4 \rangle$

$$= 4(2) - 2(5) + 3(-4) = 8 - 10 - 12 = \boxed{-14}$$

Note: The dot product results in a scalar.

Dot product is sometimes called the scalar product.

Properties of the dot product:

1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3) $c(\vec{u} \cdot \vec{v}) = c\vec{u} \cdot \vec{v} = \vec{u} \cdot c\vec{v}$
4) $\vec{0} \cdot \vec{v} = 0$
5) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Proof of #5 For $v = \langle v_1, v_2, v_3 \rangle$, then

$$\vec{v} \cdot \vec{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = v_1 v_1 + v_2 v_2 + v_3 v_3$$

$$= v_1^2 + v_2^2 + v_3^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2$$

$$= \|\vec{v}\|^2.$$

∎

Angle between two vectors

For vectors $\vec{u}$ and $\vec{v}$ with angle θ between them ($0 \le \theta \le 180^\circ$), then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}.$$

So, the angle between $\vec{u}$ and $\vec{v}$ is

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

Example:

Find the angle between vectors $\vec{u} = \langle 3,-5,7\rangle$ and $\vec{v} = \langle 1,2,3\rangle$.

$$\|\vec{u}\| = \sqrt{9+25+49} = \sqrt{83}$$

$$\|\vec{v}\| = \sqrt{1+4+9} = \sqrt{14}$$

$$\vec{u}\cdot\vec{v} = 3(1) - 5(2) + 7(3) = 3 - 10 + 21 = 14$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{14}{\sqrt{83}\,\sqrt{14}} \approx 0.4107$$

$$\theta = \cos^{-1}\left(\frac{14}{\sqrt{83}\,\sqrt{14}}\right) \approx \cos^{-1}(0.4107) \approx \boxed{65.75^\circ}$$

Useful consequence of $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

(sometimes called the alternative definition of the dot product)

Orthogonal Vectors

(if we put them together, they meet at a right angle)

> **Theorem:**
> Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u}\cdot\vec{v} = 0$.

why?

next page

Why $\vec{u} \cdot \vec{v} = 0$ for orthogonal vectors?

Use $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$

If $\vec{u} \cdot \vec{v} = 0$, then $\cos\theta = 0$, so θ must be a right angle.

Similarly, vectors are parallel if $\cos\theta = \pm 1$

Ex: Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal, parallel, or neither.

$\vec{u} = \langle -4, 6, 10 \rangle$, $\vec{v} = \langle 6, -9, 15 \rangle$

suppose $-4c = 6$ (x-components)

$c = \frac{6}{-4} = -\frac{3}{2}$.

Does $c = -\frac{3}{2}$ work with y and z-components?

$6\left(-\frac{3}{2}\right) = -\frac{18}{2} = -9$ ✓

$10\left(-\frac{3}{2}\right) = -\frac{30}{2} = -15 \neq 15$ z-component doesn't work.

So they are not parallel.

Note: $\langle -4, 6, 10 \rangle$ and $\langle 6, -9, -15 \rangle$ are parallel.

$\vec{u} \cdot \vec{v} = 72 \neq 0$. So they are not orthogonal.

$\boxed{\vec{u} \text{ and } \vec{v} \text{ are neither parallel nor orthogonal.}}$