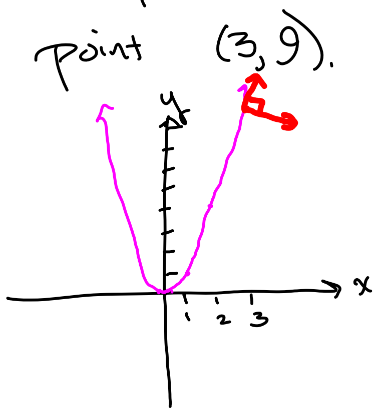
Homework Q's

Note Title 9/1/2015

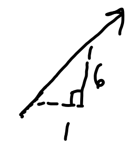
11.1 #67 Find a unit vector (a) parallel and (b) perpendicular to the graph of $f(x) = x^2$ at the point $(3, 9)$.



Find slope: $f'(x) = 2x$

$f'(3) = 2(3) = 6$

slope $= \frac{6}{1}$



Find a vector in same direction as tangent line: $\vec{v} = \langle 1, 6 \rangle$

Normalize it: $\|\vec{v}\| = \sqrt{1^2 + 6^2} = \sqrt{37}$

(a) Parallel unit vector: $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, 6 \rangle}{\sqrt{37}}$

$= \left\langle \frac{1}{\sqrt{37}}, \frac{6}{\sqrt{37}} \right\rangle$

(b) Perpendicular $\Rightarrow$ slope is opposite reciprocal

$m_\perp = -\frac{1}{6}$

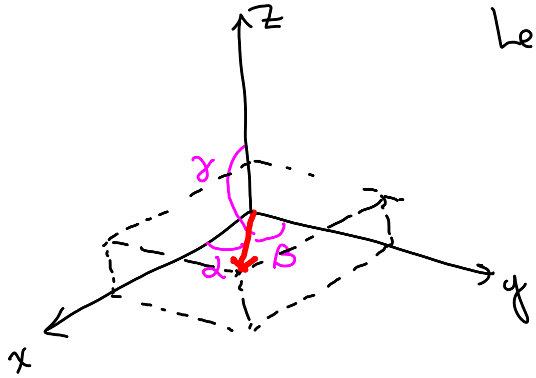
$\vec{w} = \langle 6, -1 \rangle$

$\|\vec{w}\| = \sqrt{37}$ also,

so perpendicular vector is $\frac{\vec{w}}{\|\vec{w}\|} = \left\langle \frac{6}{\sqrt{37}}, \frac{-1}{\sqrt{37}} \right\rangle$.

11.3 The Dot Product (continued)

Direction Cosines



Let α = angle between $\vec{v}$ and x-axis

β = angle between $\vec{v}$ and y-axis

γ = angle between $\vec{v}$ and z-axis

α = "alpha"

β = "Beta"

γ = "gamma"

Suppose $\vec{v} = \langle v_1, v_2, v_3 \rangle$

Direction cosines are

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_2}{\|\vec{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\vec{v}\|}$$

Why? Recall: angle θ between $\vec{u}$ and $\vec{v}$ found by

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

To find α, use vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{i}$ = unit vector in direction of x-axis

$$\cos\alpha = \frac{\vec{v}\cdot\vec{i}}{\|\vec{v}\|\|\vec{i}\|} = \frac{\langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle}{\|\vec{v}\|(1)}$$

$$= \frac{v_1 + 0 + 0}{\|\vec{v}\|} = \frac{v_1}{\|\vec{v}\|}$$

So, the direction angles α, β, γ are

$$\alpha = \cos^{-1}\left(\frac{v_1}{\|\vec{v}\|}\right), \quad \beta = \cos^{-1}\left(\frac{v_2}{\|\vec{v}\|}\right), \quad \gamma = \cos^{-1}\left(\frac{v_3}{\|\vec{v}\|}\right)$$

Note: the unit vector in the direction of $\vec{v}$ is

$$\frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{v_1}{\|\vec{v}\|}, \frac{v_2}{\|\vec{v}\|}, \frac{v_3}{\|\vec{v}\|} \right\rangle = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

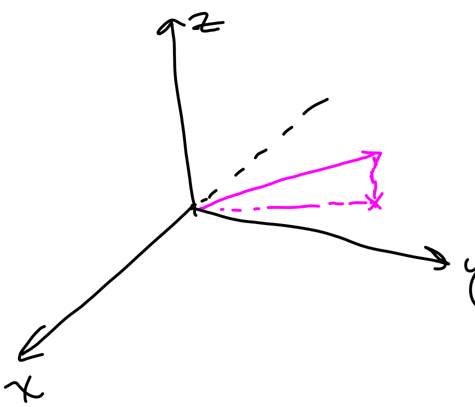
Example: Find the angles between $\vec{u} = \langle -2, 6, 1 \rangle$ and the axes. (in degrees, round to 2 decimal places)

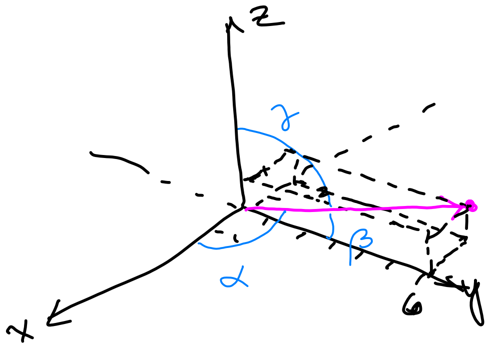
$$\|\vec{u}\| = \sqrt{41}$$

$$\cos\alpha = \frac{-2}{\sqrt{41}} \Rightarrow \alpha \approx 108.20^\circ$$

$$\cos\beta = \frac{6}{\sqrt{41}} \Rightarrow \beta \approx 20.44^\circ$$

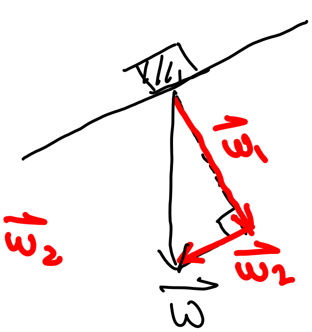
$$\cos\gamma = \frac{1}{\sqrt{41}} \Rightarrow \gamma \approx 81.02^\circ$$



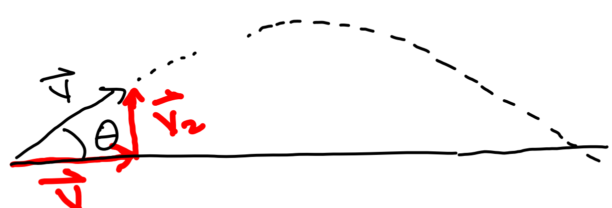


Projections and vector components:

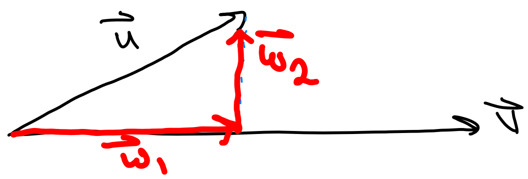
Examples:



$\vec{w} = \vec{w}_1 + \vec{w}_2$



$\vec{v} = \vec{v}_1 + \vec{v}_2$



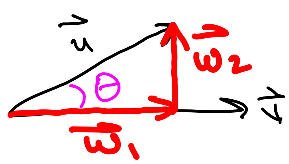
Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let $\vec{u} = \vec{w}_1 + \vec{w}_2$ where $\vec{w}_1$ is parallel to $\vec{v}$ and $\vec{w}_2$ is orthogonal to $\vec{v}$. Then,

1) $\vec{w}_1$ is called the <u>projection of $\vec{u}$ onto $\vec{v}$</u> and is denoted $\vec{w}_1 = \text{proj}_{\vec{v}}\,\vec{u}$.

2) $\vec{w}_2 = \vec{u} - \vec{w}_1$

<u>Theorem:</u> $\boxed{\vec{w}_1 = \text{proj}_{\vec{v}}\,\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}}$ ☆ Know this

Why?



$$\sin\theta = \frac{\|\vec{w}_2\|}{\|\vec{u}\|} \quad \text{and} \quad \cos\theta = \frac{\|\vec{w}_1\|}{\|\vec{u}\|}$$

$$\|\vec{w}_2\| = \|\vec{u}\|\sin\theta$$

$$\|\vec{w}_1\| = \|\vec{u}\|\cos\theta$$

$$\|\vec{w}_1\| = \cancel{\|\vec{u}\|}\left(\frac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\,\|\vec{v}\|}\right)$$

$$= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

multiply this by a unit vector in the direction of $\vec{v}$:

$$\vec{w}_1 = \|\vec{w}_1\|\left(\frac{\vec{v}}{\|\vec{v}\|}\right)$$

$$= \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)}_{\|\vec{w}_1\|}\underbrace{\left(\frac{\vec{v}}{\|\vec{v}\|}\right)}_{\text{unit vector in } \vec{v} \text{ direction}}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

Ex: Suppose $\vec{u} = \langle 3, -5, 2 \rangle$ and $\vec{v} = \langle 2, 3, -4 \rangle$.

Find $\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u}$. Also find $\vec{w}_2$, (the component of $\vec{u}$ that is orthogonal to $\vec{v}$.)

$$\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

$$= \frac{\langle 3, -5, 2 \rangle \cdot \langle 2, 3, -4 \rangle}{\| \langle 2, 3, -4 \rangle \|^2} \langle 2, 3, -4 \rangle$$

$$= \frac{6 - 15 - 8}{\left(\sqrt{4+9+16}\right)^2} \langle 2, 3, -4 \rangle$$

$$= \frac{-17}{29} \langle 2, 3, -4 \rangle$$

$$= \left\langle -\frac{34}{29}, -\frac{51}{29}, \frac{68}{29} \right\rangle$$

$$\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u} = \left\langle -\frac{34}{29}, -\frac{51}{29}, \frac{68}{29} \right\rangle$$

$$\vec{w}_2 = \vec{u} - \vec{w}_1 = \langle 3, -5, 2 \rangle - \left\langle -\frac{34}{29}, -\frac{51}{29}, \frac{68}{29} \right\rangle$$

$$= \left\langle \frac{121}{29}, \frac{-94}{29}, \frac{-10}{29} \right\rangle$$

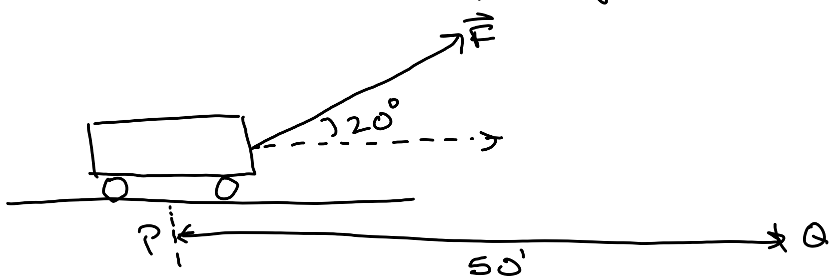
Work:

The work done by force $\vec{F}$ in moving an object from P to Q is

$$W = \vec{F} \cdot \overrightarrow{PQ}$$

$$W = \|\text{proj}_{\overrightarrow{PQ}} \vec{F}\| \, \|\overrightarrow{PQ}\|$$

Example: A force of 25 lbs is applied to a wagon at an angle of $20°$ above the horizontal. Find the work done in pulling the wagon 50 ft.



$$\vec{F} = \langle 25 \text{ lbs} \cos 20°, 25 \text{ lbs} \sin 20° \rangle$$

$$\overrightarrow{PQ} = \langle 50 \text{ ft}, 0 \rangle$$

$$W = \vec{F} \cdot \overrightarrow{PQ} = \langle 25 \text{ lb} \cos 20°, 25 \text{ lb} \sin 20° \rangle \cdot \langle 50 \text{ ft}, 0 \rangle$$

$$= (25 \text{ lb} \cos 20°)(50 \text{ ft}) + 0$$

$$= \boxed{1174.6 \text{ ft-lb}}$$