

## 5.1: Probability Rules

An *experiment* is an activity with observable results. An experiment that does not always give the same result, even under the same conditions, is called a *probability experiment*, or a *random experiment*. Repetitions of an experiment are called *trials*.

Examples: rolling dice, drawing cards, counting the number of defective radios in a sample from a shipment, using Excel to generate a random number, measuring the height of a plant, etc.

In probability theory, we'll usually use the word *experiment* to mean a *probability experiment*.

For a given experiment, we can make a list of outcomes of the experiment, called *simple events*, such that in each trial, one and only one of the simple events will occur. The set of all such simple events (outcomes) is called the *sample space*.

Any subset of the sample space is called an *event*. If such a subset contains more than one element of the sample space, we call the subset a *compound event*. If an event  $E = \emptyset$ , we call the event  $E$  an *impossible event*. If  $S$  is the sample space and  $E = S$ , then  $E$  is a *certain event*.

**Example 1:** We roll a single six-sided die. What is the sample space?

$n(S) = 36$  ✓

Roll a pair of dice:

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$n(S) = 6$$

$$\left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \\ (2,1), (2,2), \dots \\ \vdots \\ (6,1), \dots, \dots, (6,6) \end{array} \right\}$$

**Example 2:** We spin an American roulette wheel, which has pockets labeled 00, 0, 1, 2, 3, ..., 35, 36. What is the sample space?

$$S = \{00, 0, 1, 2, 3, \dots, 35, 36\}; \quad n(S) = 38$$

**Example 3:** The manager of a local cinema records the number of patrons attending a particular movie. The theater has a seating capacity of 500.

- a. What is an appropriate sample space for this experiment?

$$S = \{0, 1, 2, 3, \dots, 499, 500\} \quad n(S) = 501$$

- b. Write the event  $E$  that fewer than 50 people attend the movie.

$$E = \{0, 1, 2, 3, 4, \dots, 49\}$$

- c. Write the event  $F$  that the theater is more than half full at the movie.

$$F = \{251, 252, 253, \dots, 500\}$$

**Example 4:** An experiment consists of studying the composition of a three-child family.

- Write an appropriate sample space for this experiment if we are interested in the genders of the children in the order of their births.
- Write the event  $E$  that there are two girls and a boy in the family.
- Write the event  $F$  that the oldest child is a girl.
- Write the event  $G$  that the oldest child is a girl and the youngest is a boy.
- Write an appropriate sample space if we are only interested in the number of girls and not the order.

In these triples,  
1st child is  
listed first

b = boy, g = girl

a)  $S = \{bbb, bbq, bq b, qb b, bqq, qbq, qqb, qqq\}$

$$n(S) = 8$$

b)  $E = \{bqq, qbq, qqb\}$

c)  $F = \{qbq, qb b, qqb, qqq\}$

d)  $G = \{qb b, qqb\}$

e)  $S_2 = \{0, 1, 2, 3\}$

### Probability:

*Probability* is a way of quantifying the likelihood that an event occurs. There are three ways of assigning probabilities:

Empirical approach: Collect data, and use these data to assign a probability to each outcome.

What is the prob. a student here drives a compact car?  
Could count a sample of cars, determine the % age that are compact.

Theoretical (classical) approach: Assign probabilities using logic and reasoning.

Calculate probability of rolling a 5 on a die roll.  
~~Probability~~ Probs of getting a 10 on a dartboard.

Subjective approach: Assign probabilities based on assumptions, educated guesses, and estimates.

A vet estimates the probability that a dog survives a surgery. (or that the surgery works)

**Probability models:**

A *probability model*, or *probability assignment*, is a function that associates each outcome with a probability.

Suppose  $S = \{e_1, e_2, \dots, e_n\}$  is a sample space. It contains  $n$  simple events, or outcomes. To each outcome, we can assign a number  $P(e_i)$ , called the *probability of the event*  $e_i$ .

Rules for probability assignments:

- For each simple event, the probability of that event is a number between 0 and 1, inclusive. In other words,  $0 \leq P(e_i) \leq 1$ .
- The probabilities of all the simple events in the sample space add up to 1. In other words,  $P(e_1) + P(e_2) + \dots + P(e_n) = 1$ .

Any probability assignment that meets these conditions is called an *acceptable probability assignment*. (or a probability model)

A probability assignment that reflects the actual or expected percentage of times a simple event occurs is called *reasonable*. In other words, it is reasonable if it makes sense based on the real world.

**Example 5:** Roll a single die.

$$S = \{1, 2, 3, 4, 5, 6\}$$

A probability assignment that is both acceptable and reasonable:

$$\begin{array}{ll} P(1) = 1/6 & P(4) = 1/6 \\ P(2) = 1/6 & P(5) = 1/6 \\ P(3) = 1/6 & P(6) = 1/6 \end{array}$$

A probability assignment that is not acceptable:

$$\begin{array}{l} P(1) = 1/6 \\ P(2) = 5/6 \\ P(3) = 1/6 \end{array} \quad \text{(adds up to more than 1)}$$

A probability assignment that is acceptable but not reasonable:

$$\begin{array}{ll} P(1) = \{1/2\} & P(3) = P(4) = P(5) = P(6) = 0 \\ P(2) = \{1/2\} & \end{array}$$

**Empirical probability assignments:**

Empirical (relative frequency) probability approximation:

$P = \text{probability}$

$$P(E) \approx \frac{\text{frequency of occurrence of } E}{\text{total number of trials}} = \frac{f(E)}{n}$$

(The larger  $n$  is, the better the approximation.)

**Example 6:** Suppose a coin is flipped 1000 times, resulting in 603 heads and 397 tails. Use these results to assign empirical probabilities for the experiment of flipping that coin.

H: Heads

T: Tails

$$P(H) = \frac{603}{1000} = 0.603$$

$$P(T) = \frac{397}{1000} = 0.397$$

$$P(H) = \frac{f(H)}{n} = \frac{603}{1000}$$

**Example 7:** During the Spring 2016 semester, Lone Star College – North Harris enrolled 10,124 female students, 6,437 male students, and 10 students of unknown gender. Use these data to create a probability assignment. If a student is randomly selected, what is the probability that student is a female?

Data from

[http://www.lonestar.edu/images/Student\\_Demographics\\_Official\\_Day\\_Spring\\_2015.pdf](http://www.lonestar.edu/images/Student_Demographics_Official_Day_Spring_2015.pdf).

$$n = 10124 + 6437 + 10 = 16571$$

F: female

M: male

U: unknown

$$P(F) = \frac{f(F)}{n} = \frac{10124}{16571} \approx 0.61$$

$$P(M) = \frac{f(M)}{n} = \frac{6437}{16571} \approx 0.39$$

$$P(U) = \frac{f(U)}{n} = \frac{10}{16571} \approx 0$$

Probability Assignment

$$P(F) = 0.61$$

$$P(M) = 0.39$$

$$P(U) = 0$$

(Prob. model)

**Theoretical (classical) probability assignments:**

When assigning probabilities theoretically (using logic and reasoning instead of data), we often depend on the *equally likely assumption*. To use this assumption, we need to be working with a sample space in which all the outcomes, or simple events, are equally likely to occur (i.e., their probabilities are equal). If several choices are possible for an experiment's sample space, it is often best to choose one in which all outcomes are equally likely.

The probability of a randomly selected student being female is 0.61

Equally likely assumption:

If all events in a sample space are equally likely to occur, the probability of each is  $\frac{1}{n}$ , where  $n$  is the number of simple events in the sample space.

The equally likely assumption results in a basic principle of probability.

Basic Probability Principle:

Let  $S$  be a sample space of equally likely outcomes, and let the event  $E$  be a subset of  $S$ . Then the probability that event  $E$  occurs is

$$P(E) = \frac{n(E)}{n(S)}.$$

**Example 8:** Using the theoretical (classical) approach, create a probability assignment for the flip of a single coin.

H: heads  
T: tails

$$\begin{aligned} P(H) &= \frac{1}{2} \\ P(T) &= \frac{1}{2} \end{aligned}$$

Sample Space:  
 $S = \{H, T\}$   
They are  
equally likely

How do theoretical probabilities and empirical probabilities relate to each other?

Law of Large Numbers:

As the number of repetitions of an experiment increases, the empirical probability of an event approaches the theoretical probability of the event.

Whether the probability assignment is created empirically or theoretically, the following principles apply.

Probability of an event  $E$ :

probability model

Given a sample space  $S$  and an acceptable probability assignment, then for any event  $E$  ( $E \subset S$ ), the following rules apply:

- If  $E = \emptyset$ , then  $P(E) = 0$ .
- If  $E$  is a simple event, then  $P(E)$  is given by the original probability assignment.
- If  $E$  is a compound event, then  $P(E)$  is the sum of the probabilities of all the simple events in  $E$ .
- If  $E = S$ , then  $P(E) = P(S) = 1$ .

Steps for finding  $P(E)$ :

1. Set up an appropriate sample space.
2. Assign acceptable probabilities to each simple event.
3.  $P(E)$  is the sum of the probabilities of all the simple events in  $E$ .

Note on notation:

$E \subset S$  means that  $E$  is a subset of  $S$  (i.e., every element of  $E$  is also an element of  $S$ ).

The *empty set* is the set with no elements. The empty set is denoted  $\emptyset$ .

**Example 9:** Consider the following data from the University of Houston. Data from <http://www.uh.edu/ir/reports/facts-at-a-glance/facts-at-a-glance.pdf>.

- a. What is the probability that a randomly selected undergraduate is in the College of Business?  $S = \{\text{all undergrads who could be selected}\}$

B: student is in college of Business

$$P(B) = \frac{n(B)}{n(S)} = \frac{4885}{33404} \approx 0.146$$

- b. What is the probability that a randomly selected graduate student is in a health-related program?  $S = \{\text{all graduate students}\}$

H: student is in health-related program

$$P(H) = \frac{34 + 43 + 79}{6296} \approx 0.0247$$

## UNIVERSITY of HOUSTON

### Fall 2015 Facts

#### Student Enrollment

College	Under-Graduate	Post-Baccalaureate	Graduate	Special Professional	Total
Architecture	605	3	81		689
Business	4,885	200	1,020		6,105
Education	1,694	61	676		2,431
Engineering	3,467	292	1,312		5,071
HRM	946	8	86		1,040
Law			130	717	847
CLASS	10,608	347	1,081		12,036
NSM	3,946	284	975		5,205
Nursing	42	54	34		130
Optometry			43	393	436
Pharmacy		5	79	468	552
Social Work			405		405
Technology	5,221	142	374		5,737
Exploratory Studies	1,990	30			2,020
Total	33,404	1,426	6,296	1,578	42,704

HRM—Conrad N. Hilton College of Hotel and Restaurant Management

CLASS — The College of Liberal Arts and Social Sciences

NSM— The College of Natural Sciences and Mathematics

H: heads  
T: tails

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**Example 10:** Flip two fair coins. What is the probability of getting

- Two tails?
- At least one head?
- Exactly one head?
- Two heads or two tails?
- Three tails?

$S = \{HT, HH, TH, TT\}$   
all are equally likely.  
 $n(S) = 4$

a)  $P(TT) = \boxed{\frac{1}{4}}$

b) B: at least 1 head  
 $B = \{HT, HH, TH\}$   
 $P(B) = \frac{n(B)}{n(S)} = \boxed{\frac{3}{4}}$

c) C: exactly 1 head  
 $C = \{HT, TH\}$   
 $P(C) = \frac{n(C)}{n(S)} = \frac{2}{4} = \boxed{\frac{1}{2}}$

e) E: 3 tails  
 $P(E) = \boxed{0}$   
You could say  
that  $E = \emptyset$

Important Note: For any event  $E$ ,

$$0 \leq P(E) \leq 1.$$

If  $P(E) = 0$ , then  $E$  is an *impossible event*.

If  $P(E) = 1$ , then  $E$  is an *certain event*.

d) D: 2 heads or 2 tails  
 $D = \{HH, TT\}$   
 $P(D) = \frac{n(D)}{n(S)} = \frac{2}{4} = \boxed{\frac{1}{2}}$

**Example 11:** Draw a single card from a standard card deck. What is the probability that it is

- A spade?
- A queen?
- Black?
- Higher than a 7? (Assume aces high.)

$S = \{A, 2, \dots, 10, J, Q, K \text{ in each of 4 suits}\}$

a)  $P(\text{Spade}) = \frac{n(\text{Spade})}{n(S)} = \frac{13}{52} = \boxed{\frac{1}{4}}$

b)  $P(\text{Queen}) = \frac{n(Q)}{n(S)} = \frac{4}{52} = \boxed{\frac{1}{13}}$

c)  $P(\text{Black}) = \frac{n(\text{Black})}{n(S)} = \frac{26}{52} = \boxed{\frac{1}{2}}$

$n(S) = 13(4) = 52$

d) D: set of cards higher than 7

$D = \{8, 9, 10, J, Q, K, A\}$   
in each suit

$n(D) = 7(4) = 28$

$P(D) = \frac{n(D)}{n(S)} = \frac{28}{52} = \frac{14}{26} = \boxed{\frac{7}{13}}$

**Example 12:** Roll a single die. What is the probability of rolling

- a) a 5?
- b) a prime number?
- c) a multiple of 3?
- d) a number larger than 10?
- e) a number smaller than 10?

Sample Space:  $S = \{1, 2, 3, 4, 5, 6\}$

(a) A: a 5 is rolled

$$A = \{5\}$$

$$P(A) = \frac{n(A)}{n(S)} = \boxed{\frac{1}{6}}$$

(b) B: A prime number is rolled

$$B = \{2, 3, 5\}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \boxed{\frac{1}{2}}$$

(c) C: a multiple of 3 is rolled.

$$C = \{3, 6\}$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

(d) D: A number larger than 10 is rolled.

$$D = \emptyset \quad (\text{no such numbers in sample space})$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{0}{6} = \boxed{0}$$

(e) E: A number less than 10 is rolled

$$E = \{1, 2, 3, 4, 5, 6\}$$

$$P(E) = \frac{n(E)}{n(S)}$$

$$= \frac{6}{6}$$

$$= \boxed{1}$$

**Example 13:** Jennifer's electricity went out last night. Not having a flashlight, she had to choose a T-shirt at random in the dark. She owns 10 white T-shirts, 5 gray ones, 6 black ones and 2 red ones. What is the probability that she chose a gray T-shirt?

$$n(S) = 10 + 5 + 6 + 2 = 23$$

$$P(\text{Gray}) = \frac{n(\text{Gray})}{n(S)} = \boxed{\frac{5}{23}}$$

Note:

D is an impossible event

E is a certain event.

Jo St Su Li

**Example 14:** Suppose that Joe, Steve, Suzy, and Lisa work for the same company. The company wants to send two representatives to a particular conference and needs two to stay home and take care of the customers. All of them want to attend the conference, so they decide to put their names in a hat and draw two at random. What is the probability that Suzy and Lisa are selected?

$$S = \text{all the ways to choose 2 people} \\ = \{JoSt, JoSu, JoLi, StSu, StLi, SuLi\}$$

$$n(S) = 6$$

$$P(SuLi) = \boxed{\frac{1}{6}}$$



**Example 15:** The age distribution of Lone Star College – North Harris students during Spring 2015 is given in the table below. Data from [http://www.lonestar.edu/images/Student\\_Demographics\\_Official\\_Day\\_Spring\\_2015.pdf](http://www.lonestar.edu/images/Student_Demographics_Official_Day_Spring_2015.pdf).

Age Range	LSC-North Harris*	
	#	%
Under 20	4,183	25%
20-24	5,525	33%
25-29	2,425	15%
30-39	2,677	16%
40-49	1,179	7%
50+	581	4%
Unknown	1	0%
<b>Grand Total</b>	<b>16,571</b>	<b>100%</b>

or just look here.

What is the probability that a randomly selected student is

a) under 20 years old?

A: student is < 20.

$$P(A) = \frac{n(A)}{n(S)} = \frac{4,183}{16,571} = 0.2524 \approx \boxed{0.25}$$

b) Over 39?

B: student is > 39

$$P(B) = 0.07 + 0.04 = \boxed{0.11}$$

c) Over 60?

Not enough info to answer exactly, because the over-60 students are mixed with the 50+.

we can be sure that  $\boxed{P(C) \leq 0.04}$

**Example 16:** Suppose a family has three children. What is the probability all the children are boys? What is the probability at least one child is a boy? Assume boys and girls are equally likely.

Set up an equally likely sample space:

$$S = \{BBG, BGB, GBB, BGG, GBG, GGB, BBB, GGG\}$$

$$n(S) = 8$$

A: All kids are boys.

$$A = \{BBB\}$$

$$P(A) = \frac{n(A)}{n(S)} = \boxed{\frac{1}{8}}$$

B: at least 1 child is a boy.

$$B = \{BBG, BGB, GBB, BGG, GBG, GGB, BBB\}$$

$$n(B) = 7$$

$$P(B) = \frac{n(B)}{n(S)} = \boxed{\frac{7}{8}}$$