5.3 and 5.4: Conditional Probability, Independence and the Multiplication Rule

Conditional probability:

Consider the probability that a house will be flooded during a given year. Would you expect this probability to be different if you only considered houses that were located in a 50-year flood plain? Yes!

Example 1: Draw a single card from a standard 52-card deck.

a. What is the probability that you draw a jack?

$P(J) = \frac{4}{52} = \boxed{\frac{1}{13}}$

b. New information….Given that you drew a face card (K,Q,J), what is the probability that it is a jack?

S = set of all face cards = {K, Q, J} of each suit

$n(S) = 3(4) = 12$

$P(J) = \frac{4}{12} = \boxed{\frac{1}{3}}$

Notation: $P(A|B)$ denotes the probability of A given that B occurs.

Conditional probability definition:

The probability of A given that B occurs is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \qquad (P(B) \neq 0)$$

Prob of A given B

"The probability of the intersection divided by the probability of the given"

c. Use the conditional probability definition to determine the probability that a jack is drawn, given that the card is a face card.

S = set of 52 cards

F = set of all face cards $\Rightarrow P(F) = 12/52$

J = set of all Jacks

$F \cap J$ = {J of each suit} $\Rightarrow P(F \cap J) = \frac{4}{52}$

$P(J|F) = \frac{P(J \cap F)}{P(F)} = \frac{4/52}{12/52} = \frac{4}{52} \cdot \frac{52}{12} = \frac{4}{12} = \boxed{\frac{1}{3}}$

Equivalent Phrasing: If a red card is drawn, what is the probability that it is the ace of diamonds.

Example 2: Draw a single card from a standard 52-card deck. What is the probability of drawing the ace of diamonds given that the card is red?

S = set of 52 cards

A: draw ace of diamonds

R: draw a red card

$P(A|R) = ?$

$A \cap R = \{A\diamondsuit\}$

$P(A \cap R) = 1/52$

$P(R) = 26/52$ (There are 26 red cards)

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{1/52}{26/52} = \frac{1}{\cancel{52}} \cdot \frac{\cancel{52}}{26} = \boxed{\frac{1}{26}}$$

Example 3: In a test conducted by the U.S. Army, it was found that of 1000 new recruits, 680 men and 320 women, 57 of the men and 3 of the women were red-green color-blind. Given that a recruit selected at random from this group is red-green color-blind, what is the probability that the recruit is a male?

S = set of all recruits

M = set of male recruits

B = set of red-green color-blind recruits

$P(M|B) = ?$

$P(M \cap B) = \frac{57}{1000}$

$P(B) = \frac{60}{1000}$

$$P(M|B) = \frac{P(M \cap B)}{P(B)} = \frac{57/1000}{60/1000} = \frac{57}{1000} \cdot \frac{1000}{60} = \boxed{\frac{57}{60}}$$

Example 4: This table shows the number of adult men and women with diabetes in 2012.
http://www.cdc.gov/diabetes/pubs/statsreport14/national-diabetes-report-web.pdf

	Diabetics (in millions)	Non-diabetics (in millions)	Total
Men	15.5	98.5	114.0
Women	13.4	106.2	119.6
Total	28.9	204.7	233.6

a. What is the probability that a randomly selected adult is diabetic, given that the person is male?

$\frac{15.5}{114} \approx 0.136$

b. What is the probability that a randomly selected adult is diabetic?

$\frac{28.9}{233.6} \approx 0.124$

c. What is the probability that a randomly selected adult is a diabetic female?

F = Female

D = diabetic

S = all adults

$$\frac{n(F \cap D)}{n(S)} = \frac{13.4}{233.6} \approx 0.057$$

d) what is prob. that a randomly selected female is diabetic?

$\frac{13.4}{119.6}$

Independence of events:

Two events are said to be *independent* ~~of~~ if the outcome of one does not affect the outcome of the other. If they are not independent, then they are said to be *dependent*.

Conditional Probability Formula:

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

multiply both sides by $P(B)$:

$$P(A|B)P(B) = P(A\cap B)$$

> Independent Events:
> Events A and B are independent events if and only if:
>
> - $P(A|B) = P(A)$ or, equivalently,
> - $P(B|A) = P(B)$ or, equivalently,
> - $P(A\cap B) = P(A)P(B)$.

$P(A|B) = P(A)$ because they're independent $\Rightarrow P(A)P(B) = P(A\cap B)$

Example 5: Draw a single card from a standard deck. Show whether the following pairs of events are independent or dependent.

a. Drawing a heart and drawing a face card.
b. Drawing a king and drawing a queen.

H: hearts

F: Face cards

S = set of 52 cards

(a) $H\cap F = \{J\heartsuit, Q\heartsuit, K\heartsuit\}$

$P(H\cap F) = \frac{3}{52}$

$P(H) = \frac{13}{52} = \frac{1}{4}$

$P(F) = \frac{12}{52} = \frac{3}{13}$

Decide whether $P(H)P(F) = P(H\cap F)$ is true:

$\frac{1}{4}\left(\frac{3}{13}\right) = \frac{3}{52}$

$\frac{3}{52} = \frac{3}{52}$, so H and F are independent.

(b) K: Kings

Q: Queens

We need to decide whether $P(K)P(Q) = P(K\cap Q)$

$P(K) = \frac{4}{52} = \frac{1}{13}$

$P(Q) = \frac{4}{52} = \frac{1}{13}$

$K\cap Q = \emptyset$, so $P(K\cap Q) = 0$

$\frac{1}{13}\left(\frac{1}{13}\right) \neq 0$, so K and Q are dependent.

Example 6: Based on the data in Example 4, does diabetes seem to be independent of gender?

D = person has diabetes

M = person is male

Is $P(D|M) = P(D)$?

$P(D|M) = \frac{15.5}{114} \approx 0.136$, $P(D) = \frac{28.9}{233.6} \approx 0.124$

They seem close to being independent.

Example 7: A survey conducted found that of 2000 women, 680 were heavy smokers and 50 had emphysema. Of those who had emphysema, 42 were also heavy smokers. Using this data, determine whether the events "being a heavy smoker" and "having emphysema" were independent events.

H: is a heavy smoker

E: has emphysema

$P(H \cap E) = \frac{42}{2000}$

$P(H) = \frac{680}{2000}$

$P(E) = \frac{50}{2000}$

To test for independence, we can compare $P(H \cap E)$ to $P(H)P(E)$

$P(H)P(E) = \frac{680}{2000}\left(\frac{50}{2000}\right) = 0.0085$

$P(H \cap E) = \frac{42}{2000} = 0.021$

$P(H)P(E) \neq P(H \cap E)$, so H and E are not independent.

(Alternatively, we could compare $P(H)$ with $P(H|E)$, or $P(E)$ with $P(E|H)$)

Example 8: Suppose that a basketball player has a 78% free throw percentage, and that his the outcome of one free throw attempt does not affect his next attempt. If he attempts two free throws, what is the probability he misses both of them?

A_1 = success on first free throw

A_2 = succes on 2^{nd} free throw.

$P(A_1) = 0.78$, $P(A_2) = 0.78$

Prob he makes both free throws is $P(A_1 \cap A_2) = P(A_1)P(A_2)$
$= 0.78(0.78)$
$= 0.6084$

Prob he misses both: $P(A_1^c \cap A_2^c) = 0.22(0.22)$
$= \boxed{0.0484}$

Note:
Probability of missing 1^{st} free throw
is $1 - 0.78 = 0.22$
(because $P(E^c) = 1 - P(E)$)

Independence of more than two events:

> If $E_1, E_2, \ldots, E_n$ are independent, then
>
> $$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)P(E_2)\ldots P(E_n).$$

$P(\text{Make}) = 0.78$

$P(\text{Miss}) = 1 - 0.78 = 0.22$

Example 9: Suppose that a basketball player has a 78% free throw percentage, and that ~~his~~ the outcome of one free throw attempt does not affect his next attempt. Suppose he attempts five free throws.

a. What is the probability he makes all five?
b. What is the probability that he makes the first three, misses one, and then makes the last?
c. What is the probability that he misses at least one of the five free throws?

a) $0.78(0.78)(0.78)(0.78)(0.78) = (0.78)^5 = \boxed{0.289}$

b) $0.78(0.78)(0.78)(0.22)(0.78)$

$= (0.78)^4(0.22) = \boxed{0.0814}$

c) E = he misses at least 1

E^c = he makes all 5 → $P(E^c) = (0.78)^5 = 0.289$

$P(E) = 1 - P(E^c) \approx 1 - 0.289 = \boxed{0.711}$

Example 10: In the Iowa Democrat caucuses, coin flips are used to resolve ties. If there are six coin flips between Candidate A and Candidate B, what is the probability that Candidate A wins all six?

$P(\text{A winning Flip \#1}) = 0.5$ same for Flip # 2–#6

$P(\text{A wins all 6}) = \underbrace{0.5(0.5)(.5)(0.5)(0.5)(0.5)}_{\text{6 trials}} = (0.5)^6$

$= \boxed{0.015625}$

Example 11: A certain loudspeaker system has four components: a woofer, a midrange, a tweeter, and an electrical crossover. It has been determined that on the average 1% of the woofers, 0.8% of the midranges, 0.5% of the tweeters, and 1.5% of the crossovers are defective. Determine the probability that a randomly chosen loudspeaker is not defective. Assume that the defects in the different types of components are unrelated.

For the whole loudspeaker to be non defective, we need for all of the components (woofers, midranges, tweeters, crossovers) to be non defective.

Write the events:

W: woofer is nondefective
$P(W) = 1 - 0.01 = 0.99$

M: midrange is nondefective
$P(M) = 1 - 0.008 = 0.992$

T: Tweeter is nondefective
$P(T) = 1 - 0.005 = 0.995$

C: Crossover is nondefective
$P(C) = 1 - 0.015 = 0.985$

$P(\text{Good loudspeaker}) = P(W \cap M \cap T \cap C) = P(W)P(M)P(T)P(C)$

$= 0.99(0.992)(0.995)(0.985) \approx \boxed{0.9625}$

(Because the defects in components are independent, we can multiply the probabilities together)

The product rule (multiplication rule) for intersections of events:

Recall: The definition for conditional probability:

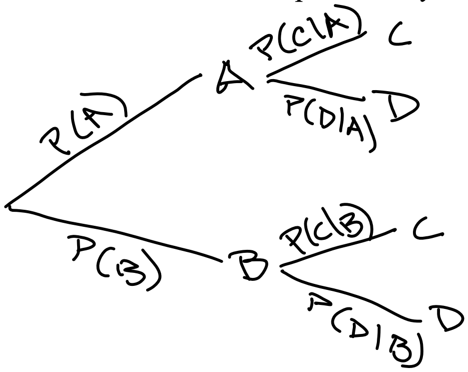
$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

mult. by $P(A)$: $P(B|A)P(A) = P(A\cap B)$

> Product rule:
>
> For events A and B with nonzero probabilities
>
> $P(A\cap B) = P(B\,|\,A)P(A)$

Probability trees:

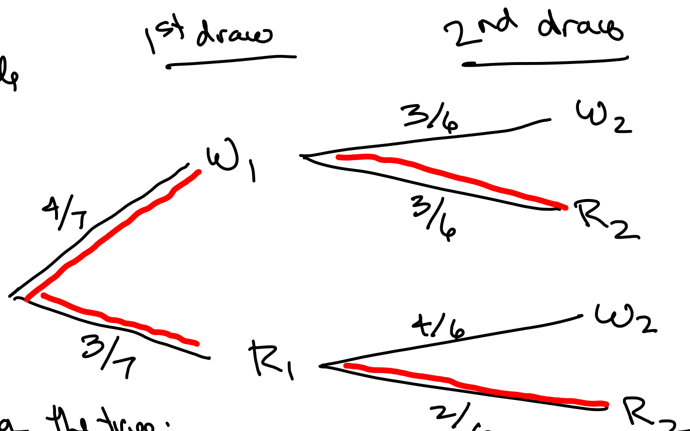
The product rule for conditional probability allows us to set up probability trees.



Example 12: A box contains 4 white marbles and 3 red marbles. Two marbles are drawn in succession without replacement. What is the probability of drawing a red marble on the second draw?

W: white marble

R: red marble



If we draw W_1, then the box only has 6 marbles left (3 white, 3 red)

If we draw R_1, then the box only has 6 marbles left (4 white, 2 red)

We multiply along the tree:

(add up all the routes that lead to R_2)

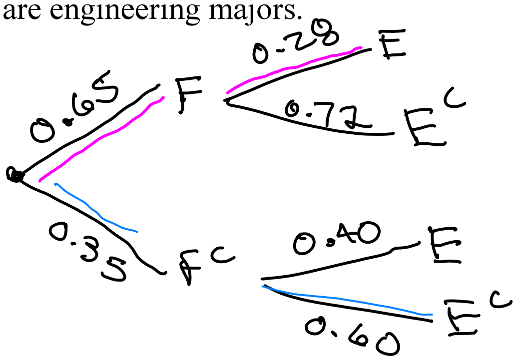
$$P(R_2) = \frac{4}{7}\left(\frac{3}{6}\right) + \frac{3}{7}\left(\frac{2}{6}\right) = \frac{12}{42} + \frac{6}{42} = \frac{18}{42} = \boxed{\frac{3}{7}}$$

F = student is Female

E = student is engineering major

Note:
student is male = F^C

Example 13: In a certain class, 65% of the students are female. 40% of the males and 28% of the females are engineering majors.



Note:

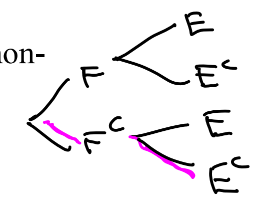
$P(E|F) = 0.28$

$P(F) = 0.65$

a. What is the probability of a randomly selected student being female and an engineering major?

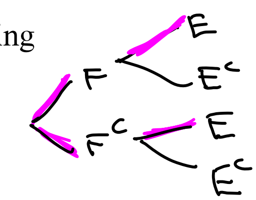
$$P(E \cap F) = P(F)P(E|F) = 0.65(0.28) = \boxed{0.182}$$

b. What is the probability of a randomly selected student being male and a non-engineering major?



$$P(F^C \cap E^C) = 0.35(0.6) = \boxed{0.21}$$

c. What is the probability of a randomly selected student being an engineering major?



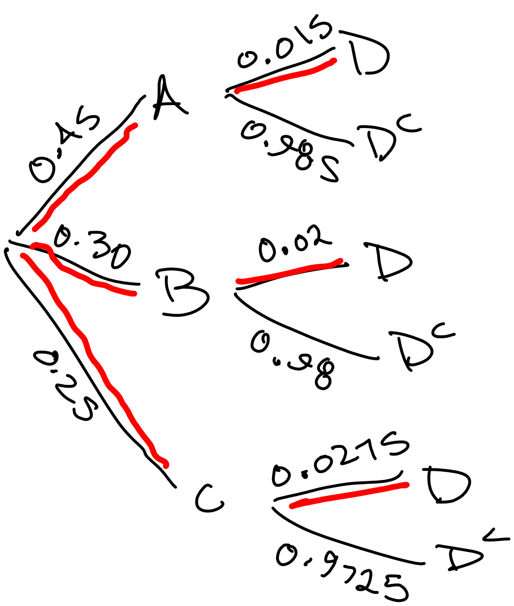
We add all the paths that include E

$$P(E) = 0.65(0.28) + 0.35(0.40) = \boxed{0.322}$$

Example 14: A certain type of camera is manufactured in three locations. Plants A, B, and C supply 45%, 30%, and 25%, respectively, of the cameras. The quality-control department of the company has determined that 1.5% of the cameras produced by plant A, 2% of the cameras produced by plant B and 2.75% of the cameras produced by plant C are defective. What is the probability that a randomly selected camera is defective?

D = camera is defective

D^C = camera is not defective



$$P(D) = 0.45(0.015) + 0.3(0.02) + 0.25(0.0275)$$

$$= \boxed{0.019625}$$