1.2: Finding Limits Graphically and Numerically

Limit of a function:

$$\lim_{x \to a} \left(f(x) \right) = L$$

Definition of a Limit:

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a} f(x) = L$$

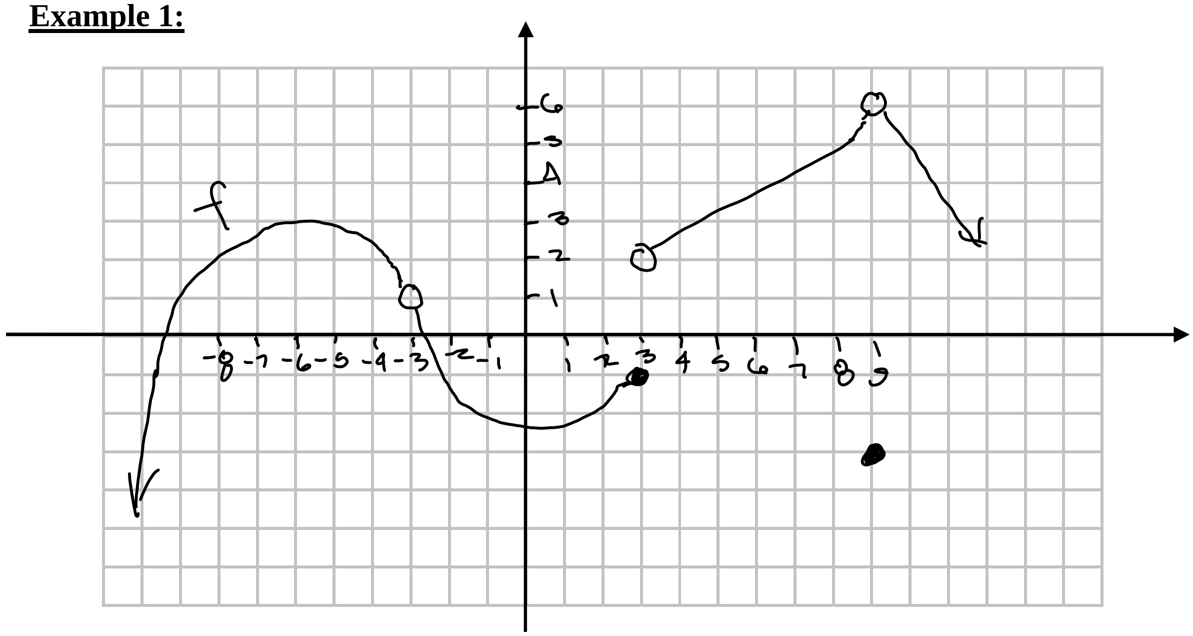
The statement above means that we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a but not equal to a.

We read this as "the limit of $f(x)$, as x approaches a, is equal to L."

Alternative notation: $f(x) \to L$ as $x \to a$. ($f(x)$ approaches L as x approaches a)

Finding limits from a graph:

Example 1:



$\lim_{x \to -3} f(x) =$ 1

$\lim_{x \to 3} f(x)$ $\nexists$ does not exist

$\lim_{x \to -8} f(x) =$ 2

$\lim_{x \to 9} f(x) =$ 6

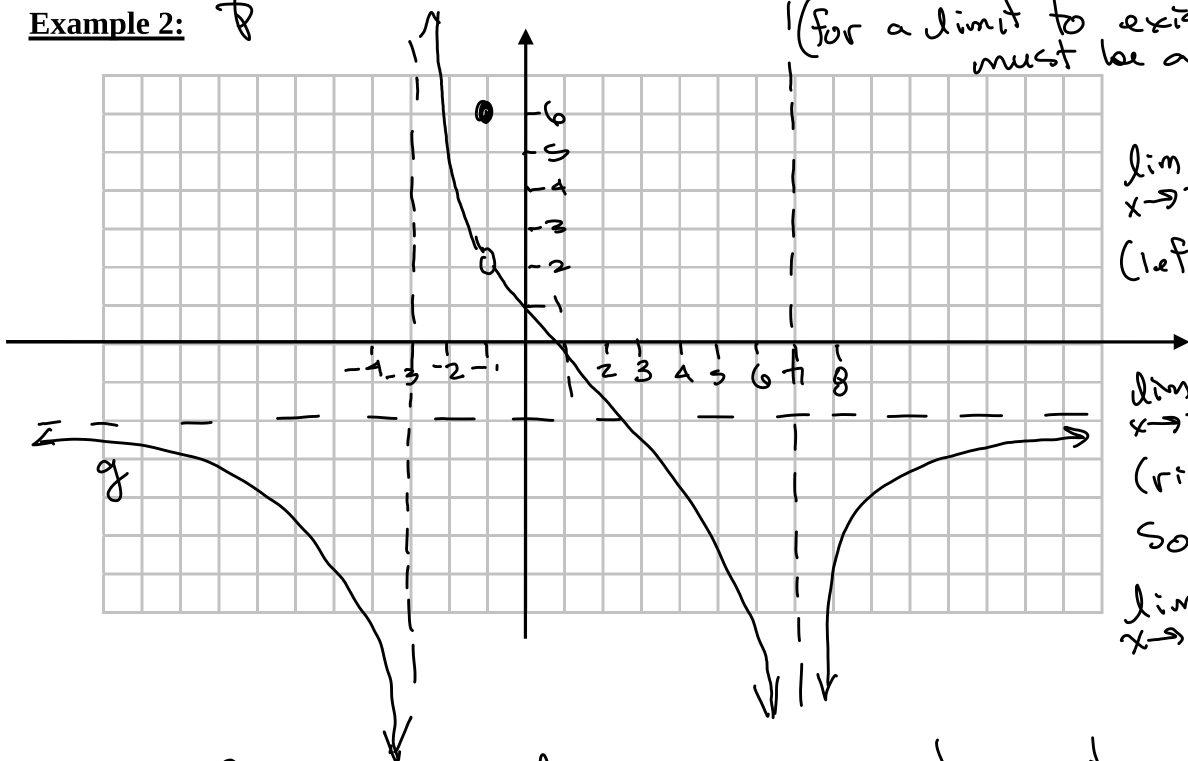
Note: $f(-3)$ does not exist.

$f(9) = -3$

Important Note: $\lim_{x\to -3} g(x)$, $\lim_{x\to 7} g(x)$, $\lim_{x\to 7^-} g(x)$, $\lim_{x\to 7^+} g(x)$

None of these limits exist!! (for a limit to exist, it must be a real number)

Example 2:



$\lim_{x\to 7^-} g(x) = -\infty$
(left-hand limit)

$\lim_{x\to 7^+} g(x) = -\infty$
(right-hand limit)

So, $\lim_{x\to 7} g(x) = -\infty$

$\lim_{x\to -1} g(x) = \underline{2}$

$g(-1) = 6$

$\lim_{x\to -3} g(x)$ does not exist.

Left-hand limit: $\lim_{x\to -3^-} g(x) = -\infty$

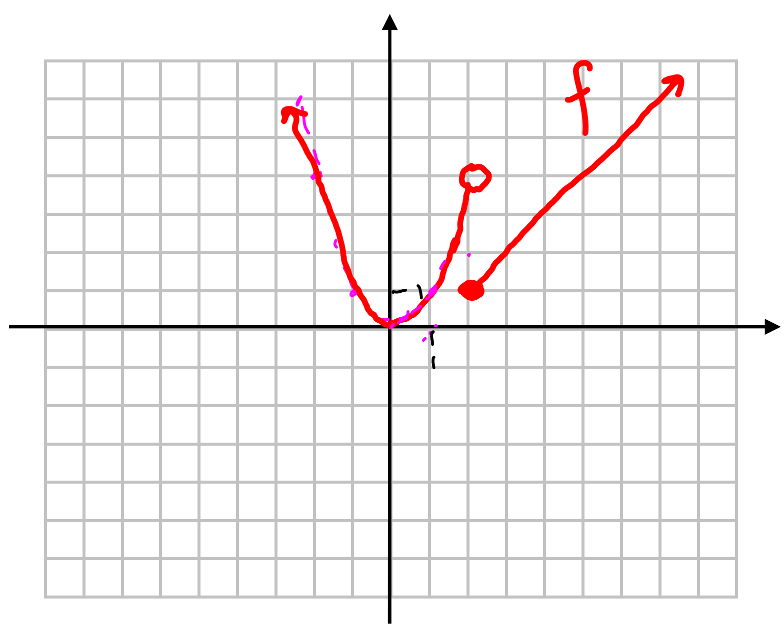
Right-hand limit: $\lim_{x\to -3^+} g(x) = \infty$

Example 3: Graph the function. Use the graph to determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to -1} f(x)$.

$$f(x) = \begin{cases} x-1 & \text{if } x \ge 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

Right of 2:
equation is: $y = x - 1$ (slope is 1, y-int is -1)

Left of 2:
equation is $y = x^2$



From graph:

$\lim_{x\to 2} f(x)$ does not exist.

$\lim_{x\to -1} f(x) = 1$

Finding limits numerically:

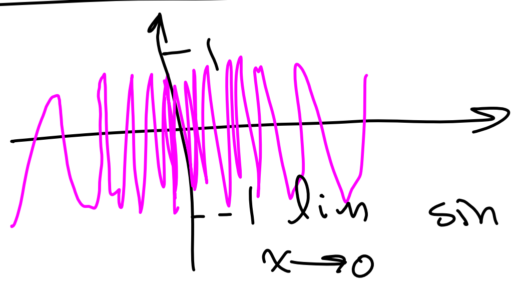
Example 4: For the function $f(x)=\dfrac{x-5}{x^2-25}$, make a table of function values corresponding to values of x near 5.

Use the table to estimate the value of $\lim\limits_{x\to 5}\dfrac{x-5}{x^2-25}$.

x	$f(x)=\dfrac{x-5}{x^2-25}$
4.8	0.10204
4.9	0.10101
4.95	0.1005
4.99	0.1001001001
4.999	0.100010001
4.9999	0.10000100001
5.2	0.9804
5.1	0.09900990099
5.05	0.099502
5.01	0.0999000999
5.001	0.09999001
5.0001	0.09999900001
5.00001	0.0999999

From Table, it appears that

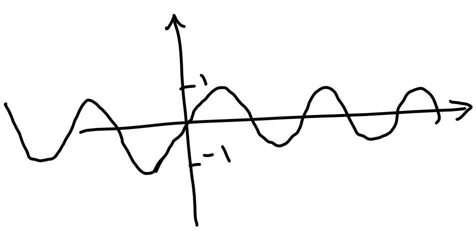
$$\lim_{x\to 5}\frac{x-5}{x^2-25}=0.1$$



$\lim\limits_{x\to 0}\sin\left(\frac{1}{x}\right)$

Example 5: Make a table of values and use it to estimate $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$.

Note: $\lim\limits_{x\to 0}\sin(x)=0$



As $x\to 0$, what happens to $\frac{1}{x}$?

Left of 0:

As $x\to 0^-$, the x's are negative tiny numbers, reciprocals are neg. huge numbers

As $x\to 0^-$, $\frac{1}{x}\to\frac{1}{-\text{tiny}}\to -\text{huge}$

From right: As $x\to 0^+$, $\frac{1}{x}\to\frac{1}{+\text{tiny}}\to +\text{huge}$

So, From left: as $x\to 0^-$, $\sin\left(\frac{1}{x}\right)\to\sin\left(\frac{1}{-\text{tiny}}\right)\to\sin(-\text{huge})$ oscillates between -1 and 1

From right: as $x\to 0^+$, $\sin\left(\frac{1}{x}\right)\to\sin\left(\frac{1}{+\text{tiny}}\right)\to\sin(+\text{huge})$ still oscillates between -1 and 1

$\boxed{\text{So, } \lim_{x\to 0}\sin(1/x) \text{ Does not exist.}}$

<u>Common reasons $\lim_{x \to c} f(x)$ may not exist:</u>

1. $f(x)$ approaches a different value when approached from the left of *c*, compared to when approached from the right of *c*.

2. $f(x)$ increases or decreases without bound as *x* approaches *c*.

3. $f(x)$ oscillates between two values as *x* approaches *c*.

The formal (epsilon-delta) definition of a limit:

<u>Definition</u>:

Let *f* be a function defined on some open interval that contains the number *a*, except possibly at *a* itself. Then

epsilon ϵ delta δ

$$\lim_{x \to a} f(x) = L$$

$\epsilon > 0$ $\delta > 0$

if for every number $\varepsilon > 0$, there is a number $\delta > 0$ such that

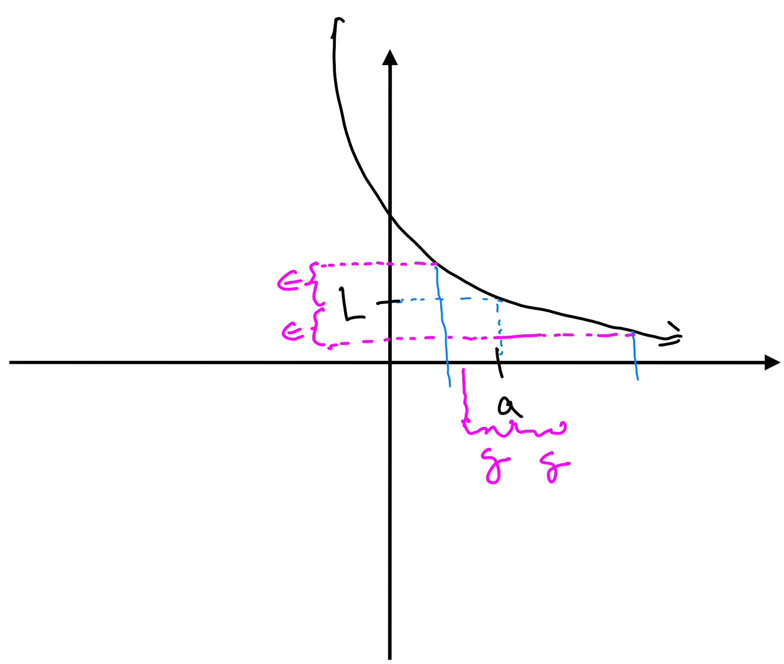
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

↳ x within δ of a $\Longrightarrow$ $f(x)$ within ϵ of L

<u>Example 6:</u>

$\lim_{x \to a} f(x) = L$

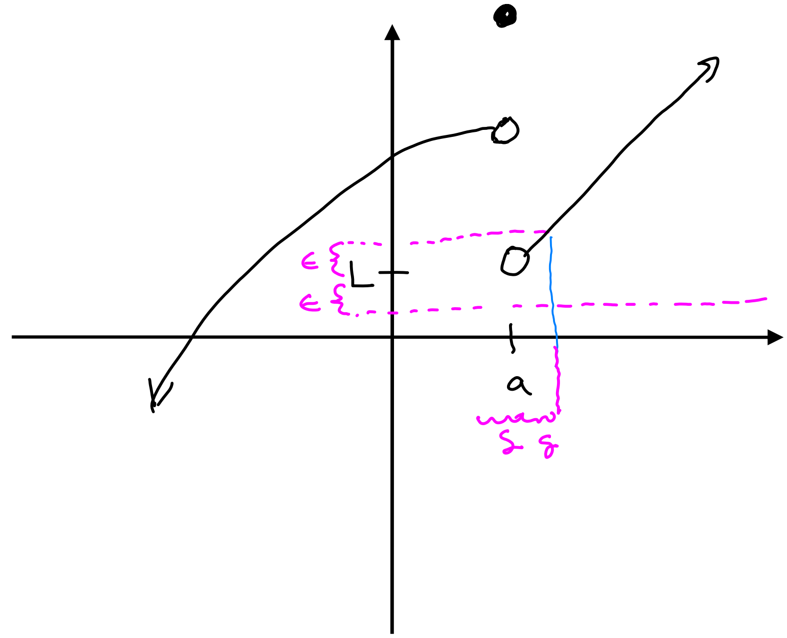
$\epsilon > 0$ is any arbitrary positive number (we think of it as being <u>small</u>)



Example 7:

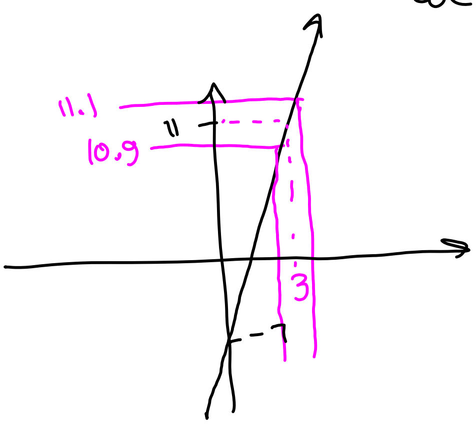
Here,

$\lim_{x \to a} f(x)$ does not exist.



Example 8: How close to 3 must we take x so that $6x-7$ is within 0.1 of 11?

Here, we think of 11 as L and we think of 3 as a.



$6x-7=11.1$

$6x=18.1$

$x=\frac{18.1}{6}=3.01\overline{6}$

$6x-7=10.9$

$6x=17.9$

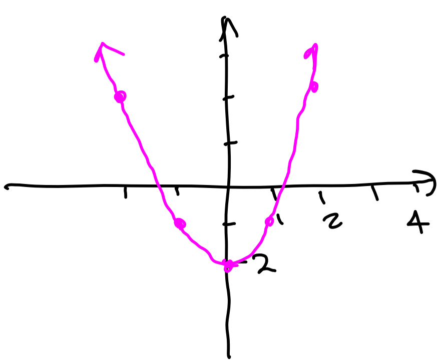
$x=\frac{17.9}{6}$

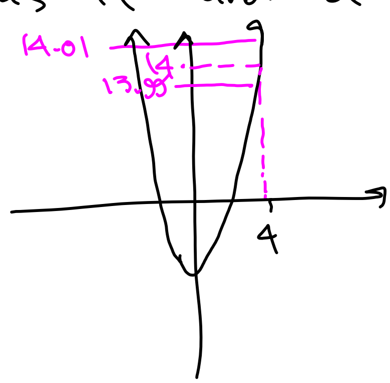
$\approx 2.98\overline{3}$

$\boxed{\text{So, } x \text{ must be within } 0.01\overline{6} \text{ of } 3. \quad (\text{Choose } \delta < 0.01\overline{6})}$

Example 9: How close to 4 must we take x so that x^2-2 is within 0.01 of 14?

Here, think of L as 14 and a as 4.





$14.01 = x^2-2$

$16.01 = x^2$

$x \approx \pm 4.001249805$

choose +

$13.99 = x^2-2$

$15.99 = x^2$

$x \approx 3.998749805$

$4-x \approx 0.0012501954$

Choose x within 0.001249805 of 4

(Choose $\delta < 0.001249805$)

$\lim_{x \to 4} (2x-5) = 3$

Example 10: Prove that $\lim_{x \to 4}(2x-5) = 3$ using the definition of a limit.

Scratchwork:

Let $\epsilon > 0$.
We want to show

$\left|\underbrace{(2x-5)-3}_{f(x)-L}\right| < \epsilon$

$|2x-8| < \epsilon$

$|2(x-4)| < \epsilon$

$|2||x-4| < \epsilon$

$2|x-4| < \epsilon$

$|x-4| < \frac{\epsilon}{2}$

Choose $\delta = \frac{\epsilon}{2}$.

Proof: Let $\epsilon > 0$. Then choose $\delta = \frac{\epsilon}{2}$.

Suppose that $0 < |x-4| < \delta$.

(We now need to show that $|(2x-5)-3| < \epsilon$.)

$|(2x-5)-3| = |2x-8| = |2(x-4)| = |2||x-4|$

$= 2|x-4| < 2\delta = 2\left(\frac{\epsilon}{2}\right) = \epsilon.$ ▨

Note: Property of absolute values:

$|AB| = |A||B|$

Example 11: Prove that $\lim_{x \to -3}(5x+1) = -14$ using the definition of a limit.

$\lim_{x \to -3} (5x+1) = -14$

Scratchwork

(done in class)

Show

$|(5x+1)-(-14)| < \epsilon$

$|5x+15| < \epsilon$

$|5(x+3)| < \epsilon$

$5|\underbrace{x+3}_{x-a}| < \epsilon.$

$|x+3| < \frac{\epsilon}{5}$

Choose $\delta = \frac{\epsilon}{5}$

(could choose any $\delta < \frac{\epsilon}{5}$)

Proof: Let $\epsilon > 0$. Then choose $\delta = \frac{\epsilon}{5}$.

Suppose $0 < |x-(-3)| < \delta$.

(we must show that $|(5x+1)-(-14)| < \epsilon$.)

$|(5x+1)-(-14)| = |5x+1+14| = |5x+15|$

$= 5|x+3| = 5|x-(-3)| < 5\delta = 5\left(\frac{\epsilon}{5}\right)$

$= \epsilon.$ ▨

Example 12: Prove that $\lim_{x\to 3} x^2 = 9$ using the definition of a limit.

Skip the proofs for quadratic functions.

Example 13: Prove that $\lim_{x\to 2}(x^2 - x + 6) = 8$ using the definition of a limit.