1.3: Evaluating Limits Analytically

Some basic limits:

Example 1: Determine $\lim\limits_{x \to a} x$.

From graph, $\lim\limits_{x \to a} x = a$

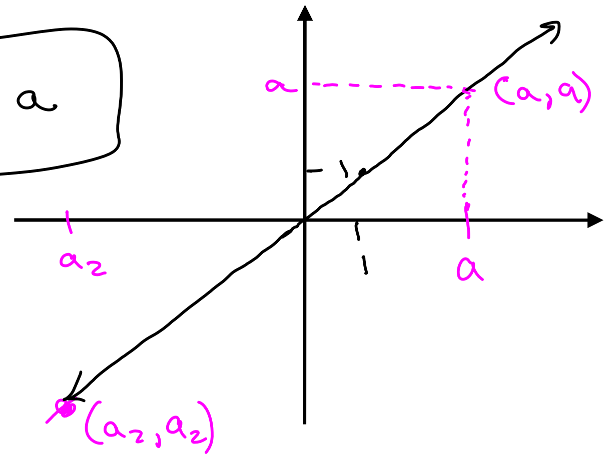
$f(x) = x$

so $y = x$

$y = 1x + 0$

slope: $m = \frac{1}{1}$

y-intercept: 0

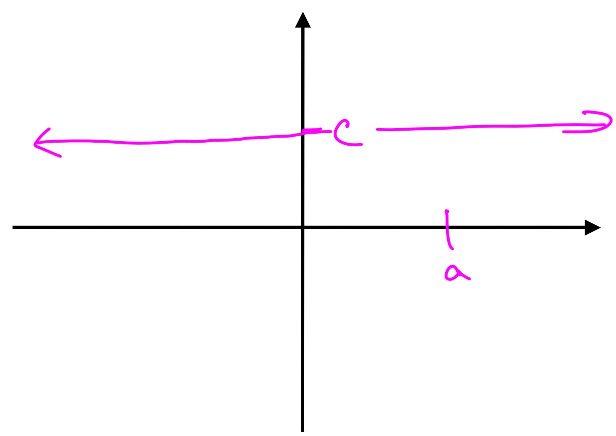


Example 2: Determine $\lim\limits_{x \to a} c$.

$f(x) = c$ so $y = c$

From graph,

$\lim\limits_{x \to a} (c) = c$



Laws (or properties) of limits:

<u>Limit Laws</u>:

Suppose that the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ if c is a constant

4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a}[f(x)]^{n/p} = [\lim_{x\to a} f(x)]^{n/p}$, if n and p are integers with no common factor, $p \neq 0$, and provided that $[\lim_{x\to a} f(x)]^{n/p}$ is a real number.

Combining the facts that $\lim_{x\to a} c = c$, $\lim_{x\to a} x = a$, and the limit laws shows that $\lim_{x\to a} x^n = a^n$ where n is a positive integer. This lets us use direct substitution for evaluating limits of polynomials.

<u>Direct Substitution Property</u>:

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

<u>Example 3:</u> Determine $\lim_{x\to 3}(4x^2 - 2x + 1)$.

$$\lim_{x\to 3}(4x^2 - 2x+1) = 4(3)^2 - 2(3)+1 = 36 - 6 + 1 = \boxed{31}$$

<u>Example 4:</u> Determine $\lim_{x\to -2} \dfrac{2x^2 - 6x + 5}{x-3}$.

$$\lim_{x\to -2} \frac{2x^2-6x+5}{x-3} = \frac{2(-2)^2 - 6(-2)+5}{-2-3} = \frac{8+12+5}{-5} = \frac{25}{-5} = \boxed{-5}$$

Example 5: Determine $\lim_{x\to 2}\sqrt[3]{4x-x^4}$.

$$\lim_{x\to 2}\sqrt[3]{4x-x^4} = \sqrt[3]{4(2)-2^4} = \sqrt[3]{8-16} = \sqrt[3]{-8}$$
$$= \boxed{-2}$$

Limit Law #6 (used in the previous example) is a special case of the following theorem:

> Limit of a Composite Function
>
> If f and g are functions such that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$
>
> $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$

Example 6: Determine $\lim_{x\to 4}\frac{x^2+x-20}{x^2-7x+12}$.

Direct substitution:

$$\frac{4^2+4-20}{4^2-7(4)+12} = \frac{20-20}{16-28+12}$$

$\Rightarrow \frac{0}{0}$ $\neq$ undefined!

This is called an indeterminate form for a limit.

Factor it first:

$$\lim_{x\to 4}\frac{x^2+x-20}{x^2-7x+12} = \lim_{x\to 4}\frac{(x+5)\cancel{(x-4)}}{(x-3)\cancel{(x-4)}}$$

$$= \lim_{x\to 4}\left(\frac{x+5}{x-3}\right) = \frac{4+5}{4-3} = \frac{9}{1}$$
$$= \boxed{9}$$

Example 7: Determine $\lim_{x\to -2}\frac{\sqrt{x+3}-1}{x+2}$.

Direct Substitution:

$$\frac{\sqrt{-2+3}-1}{-2+2} \Rightarrow \frac{0}{0}$$

indeterminate

Note: we use $(a-b)(a+b) = a^2-b^2$

$$\lim_{x\to -2}\frac{\sqrt{x+3}-1}{x+2}$$

$$= \lim_{x\to -2}\frac{\sqrt{x+3}-1}{x+2}\left(\frac{\sqrt{x+3}+1}{\sqrt{x+3}+1}\right)$$

$$= \lim_{x\to -2}\frac{x+3-1}{(x+2)(\sqrt{x+3}+1)} = \lim_{x\to -2}\frac{\cancel{x+2}}{\cancel{(x+2)}(\sqrt{x+3}+1)}$$

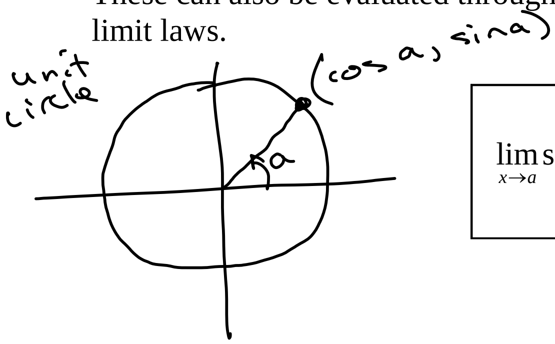
$$= \lim_{x\to -2}\left(\frac{1}{\sqrt{x+3}+1}\right) = \frac{1}{\sqrt{-2+3}+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

<u>Example 8:</u> Determine $\lim_{x\to 3}\frac{x+4}{x-3}$.

skip for now.

Limits of trigonometric functions:

These can also be evaluated through direct substitution, thanks to the two limits below, along with the limit laws.



$$\lim_{x\to a}\sin x = \sin a \qquad \lim_{x\to a}\cos x = \cos a$$

so (we can use direct substitution with trig fcns too)

<u>Example 9:</u> Determine $\lim_{x\to\frac{\pi}{3}}\tan x$.

$$\lim_{x\to\frac{\pi}{3}}(\tan x) = \lim_{x\to\frac{\pi}{3}}\frac{\sin x}{\cos x} = \frac{\sin(\pi/3)}{\cos(\pi/3)}$$

$$= \frac{\sqrt{3}/2}{1/2} = \boxed{\sqrt{3}}$$

<u>Example 10:</u> Determine $\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos(2x)}$

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos(2x)}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\cancel{\cos x-\sin x}}{(\cos x+\sin x)\cancel{(\cos x-\sin x)}}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x+\sin x} = \frac{1}{\cos\frac{\pi}{4}+\sin\frac{\pi}{4}}$$

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Direct Sub

$$\frac{\cos\frac{\pi}{4}-\sin\frac{\pi}{4}}{\cos\left(\frac{2\pi}{4}\right)}$$

$\Rightarrow \frac{0}{0}$ indeterminate form

Double Angle Identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

Pythagorean Identity

$\cos^2\theta+\sin^2\theta = 1$

Prev. ex cont'd:

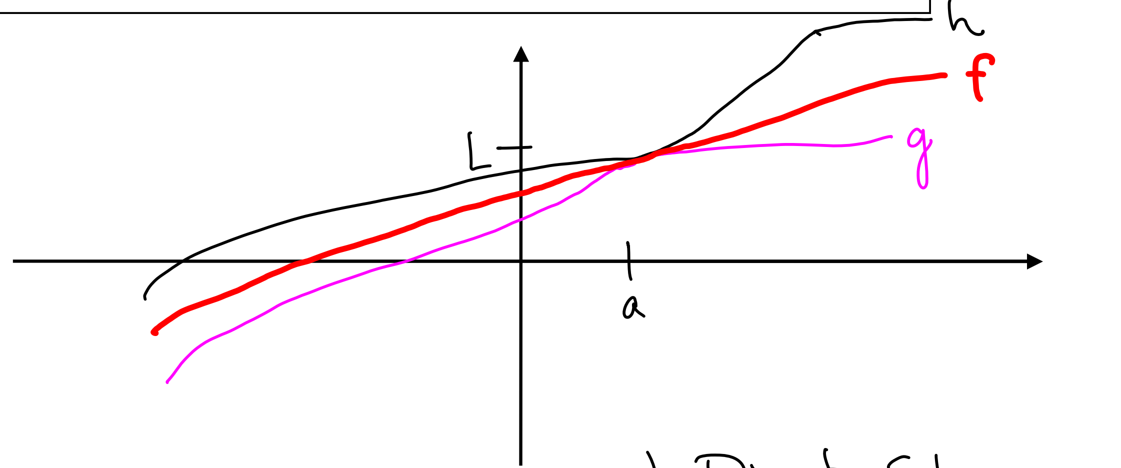
$$= \frac{1}{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}} = \frac{1}{\frac{2\sqrt{2}}{2}} = \boxed{\frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{2}}$$

The Squeeze (or Sandwich or Pinching) Theorem:

If $g(x) \le f(x) \le h(x)$ for all x in some open interval containing ~~near c,~~ a except possibly at $\cancel{x}$ a itself.
If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then

$\cancel{\lim_{x\to a} h(x) = L}$.

$$\lim_{x\to a} f(x) = L$$



Direct Sub

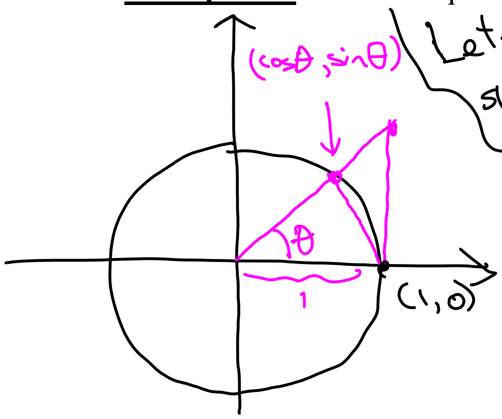
$\frac{\sin\theta}{\theta} \Rightarrow \frac{0}{0}$

indeterminate.

Example 11: Use the Squeeze Theorem to show that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

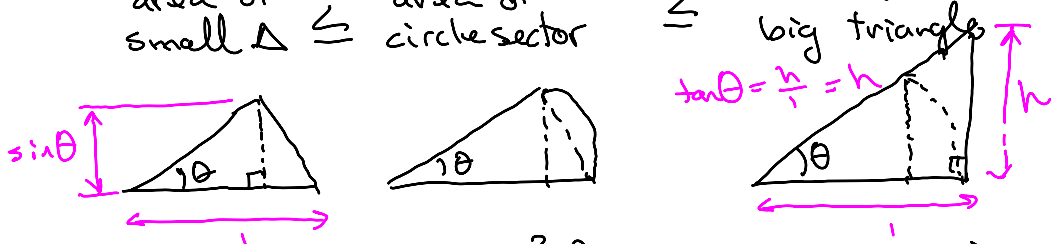
Lets use θ instead:

show $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.



area of small Δ $\le$ area of circle sector $\le$ area of big triangle

$\tan\theta = \frac{h}{1} = h$



Sector Area

$\frac{A_{sector}}{A_{circle}} = \frac{\theta}{2\pi}$

$\frac{A_{sector}}{\pi r^2} = \frac{\theta}{2\pi}$

$A_{sector} = \frac{\theta}{2\pi}(\pi r^2)$

$= \frac{\theta}{2} r^2 = \frac{1}{2} r^2\theta$

$$\frac{1}{2}(1)(\sin\theta) \le \frac{1}{2}(1)^2\theta \le \frac{1}{2}(1)(\tan\theta)$$

$$\frac{1}{2}\sin\theta \le \frac{1}{2}\theta \le \frac{1}{2}\tan\theta$$

multiply all 3 sides by 2:

$$\sin\theta \le \theta \le \tan\theta$$

Divide by $\sin\theta$:

$$\frac{\sin\theta}{\sin\theta} \le \frac{\theta}{\sin\theta} \le \frac{\tan\theta}{\sin\theta}$$

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$$1 \le \frac{\theta}{\sin\theta} \le \frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta}$$

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

Take reciprocals, changing direction of inequality signs:

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$

Note: $\lim_{\theta \to 0} (\cos\theta) = 1$

Also $\lim_{\theta \to 0} (1) = 1$.

Therefore, $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

(for positive θ)

What if θ is negative?

Note that $\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta}$ (opposite angle identity)

$= \frac{\sin\theta}{\theta}$ (same as before)

Opp. angle identities:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

Therefore from Sandwich Theorem,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Two important limits:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)=0$$

<u>Note</u>: This means that $\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1$, $\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)=0$, and $\lim_{x\to 0}\left(\frac{x}{1-\cos x}\right)$ does not exist.

<u>Example 12:</u> $\lim_{x\to 0}\left(\frac{\cos x\tan x}{x}\right)$

Direct Sub:

$$\frac{\cos(0)\tan(0)}{0} \Rightarrow \frac{1(0)}{0} \Rightarrow \frac{0}{0}$$

indeterminate

$$\lim_{x\to 0}\left(\frac{\cos x \tan x}{x}\right) = \lim_{x\to 0}\left(\frac{\cancel{\cos x}\left(\frac{\sin x}{\cancel{\cos x}}\right)}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \boxed{1}$$

<u>Example 13:</u> $\lim_{x\to \frac{\pi}{2}}\left(\frac{\cos x}{\cot x}\right)$

Direct sub:

$$\frac{\cos\frac{\pi}{2}}{\frac{\cos\frac{\pi}{2}}{\sin\frac{\pi}{2}}} \Rightarrow \frac{0}{0/1} \Rightarrow \frac{0}{0}$$

indeterminate

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos x}{\frac{\cos x}{\sin x}}\right) = \lim_{x\to\frac{\pi}{2}}\left(\cancel{\cos x}\cdot\frac{\sin x}{\cancel{\cos x}}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}(\sin x) = \sin\frac{\pi}{2} = \boxed{1}$$

<u>Example 14:</u> $\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)$

We want to rearrange to have $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$

all must match (could be $5x$)

$$\lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right) = \lim_{x\to 0}\left(\frac{1-\cos(5x)}{x}\right)\left(\frac{5}{5}\right)$$

$$= \frac{5}{1}\lim_{x\to 0}\left(\frac{1-\cos(5x)}{5x}\right).$$

As $x\to 0$, $5x\to 0$ also (because $\lim_{x\to 0}(5x)=0$)

$$= 5\lim_{5x\to 0}\left(\frac{1-\cos(5x)}{5x}\right) = 5(0) = \boxed{0}$$

We want to use $\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right) = 1$

Example 15: $\lim_{x\to 0}\left(\frac{\sin(3x)}{7x}\right)$

$$\lim_{x\to 0}\left(\frac{\sin(3x)}{7x}\right) = \frac{1}{7}\lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right) = \frac{1}{7}\lim_{x\to 0}\left(\frac{\sin(3x)}{x}\right)\left(\frac{3}{3}\right)$$

$$= \frac{1}{7}\cdot\frac{3}{1}\lim_{x\to 0}\left(\frac{\sin(3x)}{3x}\right) = \frac{3}{7}\lim_{3x\to 0}\frac{\sin(3x)}{3x}$$

$$= \frac{3}{7}(1) = \boxed{\frac{3}{7}}$$

Example 16: $\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)}$

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0}\left[\frac{\sin(3x)}{\sin(2x)}\left(\frac{x}{x}\right)\right] = \lim_{x\to 0}\left[\frac{\sin 3x}{x}\left(\frac{x}{\sin 2x}\right)\right]$$

$$= \lim_{x\to 0}\left[\left(\frac{\sin 3x}{x}\right)\left(\frac{3}{3}\right)\left(\frac{x}{\sin 2x}\right)\left(\frac{2}{2}\right)\right] = \frac{3}{1}\cdot\frac{1}{2}\lim_{x\to 0}\left[\left(\frac{\sin 3x}{3x}\right)\left(\frac{2x}{\sin 2x}\right)\right]$$

$$= \frac{3}{2}\left[\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\lim_{x\to 0}\left(\frac{2x}{\sin 2x}\right)\right]$$

$$= \frac{3}{2}\left[\lim_{3x\to 0}\left(\frac{\sin 3x}{3x}\right)\lim_{2x\to 0}\left(\frac{2x}{\sin 2x}\right)\right]$$

$$= \frac{3}{2}\left[(1)(1)\right] = \boxed{\frac{3}{2}}$$

Example 17: $\lim_{x\to 0}\frac{x^3}{\sin^3(4x)}$

Example 18: $\lim_{x\to 0}\frac{\cot 3x}{\csc 8x}$