1.4: Continuity and One-Sided Limits

One-Sided Limits: (for x an interior point of the domain)

$\lim_{x \to a^-} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the left.

$\lim_{x \to a^-} f(x) = L$

$\lim_{x \to a^+} f(x) = L$ means that $f(x)$ approaches L as x approaches a from the right.

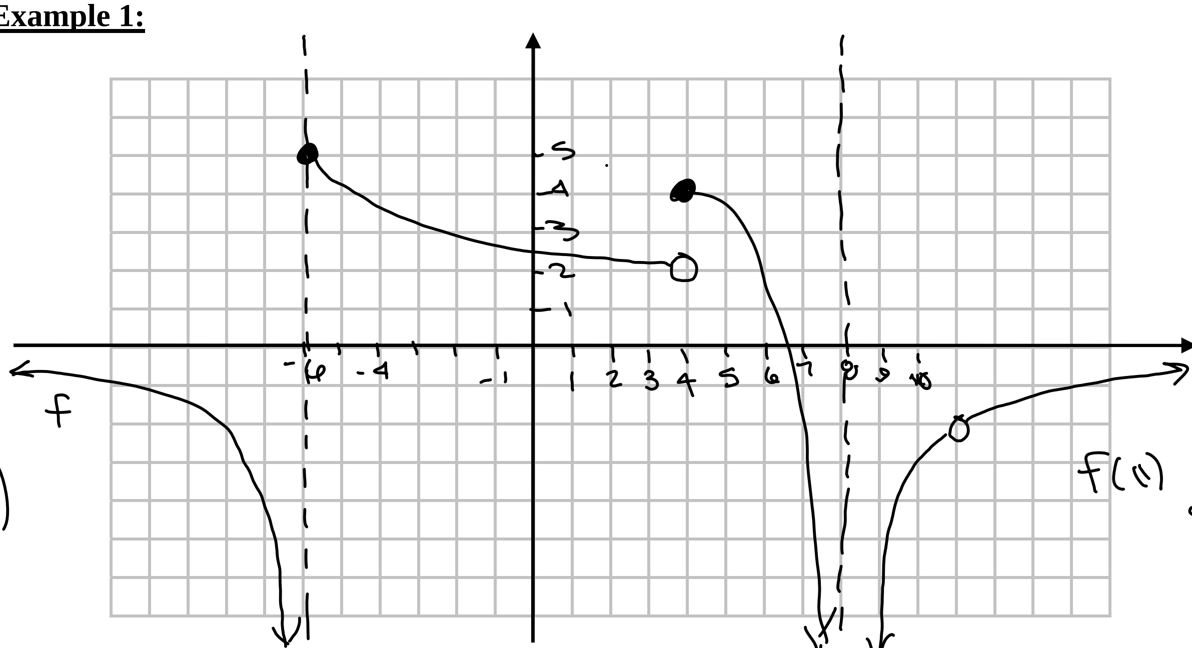
$\lim_{x \to a^+} f(x) = L$

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

iff = if and only if

$\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Example 1:



(does not exist)

f(11) does not exist

$\lim_{x \to -6^-} f(x) = -\infty$

$\lim_{x \to -6^+} f(x) = 5$

$\lim_{x \to -6} f(x)$ does not exist.

$\lim_{x \to 8^-} f(x) = -\infty$ (does not exist)

$\lim_{x \to 8^+} f(x) = -\infty$ (does not exist)

$\lim_{x \to 8} f(x) = -\infty$ (does not exist)

$\lim_{x \to 4^-} f(x) =$ 2

$\lim_{x \to 4^+} f(x) =$ 4

$\lim_{x \to 4} f(x)$ does not exist.

$\lim_{x \to 11^-} f(x) = -2$

$\lim_{x \to 11^+} f(x) = -2$

$\lim_{x \to 11} f(x) = -2$

$\lim_{x \to 14^-} f(x) = -1$

$\lim_{x \to 14^+} f(x) = -1$

$\lim_{x \to 14} f(x) = -1$

Example 2: Determine $\lim_{x\to1^-} f(x)$, $\lim_{x\to1^+} f(x)$, and $\lim_{x\to1} f(x)$, and $\lim_{x\to 5} f(x)$.

1^- $\quad 1^+ \quad 1$

$\lim_{x\to -8^-} f(x)$

$$f(x)=\begin{cases} x^2 & \text{if } x\le 1 \leftarrow \text{left of 1} \\ x-3 & \text{if } x>1 \leftarrow \text{right of 1}\end{cases}.$$

$\lim_{x\to5} f(x) = \lim_{x\to5}(x-3) = 5-3 = \boxed{2}$

$\lim_{x\to-8^-} f(x) = \lim_{x\to-8^-} (-8)^2 = \boxed{64}$

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2) = (1)^2 = \boxed{1.}$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x-3) = 1-3 = \boxed{-2}$

$\lim_{x\to1} f(x)$ does not exist.

(because the left-hand and right-hand limits don't match)

Example 3: Determine $\lim_{x\to-2^-} f(x)$, $\lim_{x\to-2^+} f(x)$, and $\lim_{x\to-2} f(x)$.

$$f(x)=\begin{cases} 1-x^3 & \text{if } x\le -2 \leftarrow \text{left of } -2 \\ 7-x & \text{if } x>-2 \leftarrow \text{right of } -2\end{cases}.$$

$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^-} (1-x^3) = 1-(-2)^3 = 1-(-8) = \boxed{9}$

$\lim_{x\to-2^+} f(x) = \lim_{x\to-2^+} (7-x) = 7-(-2) = 7+2 = \boxed{9}$

$\lim_{x\to-2} f(x) = \boxed{9}$

Note: $f(-2) = 1-(-2)^3 = 1-(-8) = 1+8 = \boxed{9}$

What is definition of $|x|$? On number line, $|x|$ is the distance from x to 0.

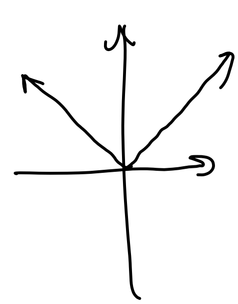
Example 4: Determine $\lim_{x\to0} |x|$, if it exists.

$$f(x)=|x|=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0\end{cases}$$

$\lim_{x\to0^-} |x| = \lim_{x\to0^-} (-x) = -0 = 0$

$\lim_{x\to0^+} |x| = \lim_{x\to0^+} (x) = 0$

so, $\lim_{x\to0} |x| = \boxed{0}$

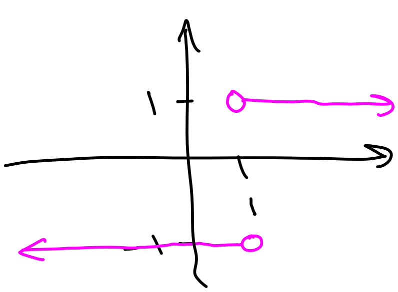


Example 5: Determine $\lim\limits_{x\to 1} f(x)$ and $\lim\limits_{x\to -2} f(x)$, where $f(x)=\begin{cases}\sqrt{5-x} & \text{if } x\ge 1\\ 3x^2-7 & \text{if } x<1\end{cases}$.

Example 6: Determine $\lim\limits_{x\to 0} f(x)$, where $f(x)=\begin{cases}\cos x & \text{if } x<0\\ 2 & \text{if } x=0\\ \sqrt{x+1} & \text{if } x>0\end{cases}$.

Example 7: Determine $\lim\limits_{x\to 1}\dfrac{|x-1|}{x-1}$.

$$\frac{|x-1|}{x-1}=\begin{cases}\dfrac{x-1}{x-1}=1 & \text{if } x-1>0 \\ & (\text{so } x>1) \\ \dfrac{-(x-1)}{x-1}=-1 & \text{if } x-1<0 \\ & (\text{so } x<1)\end{cases}$$

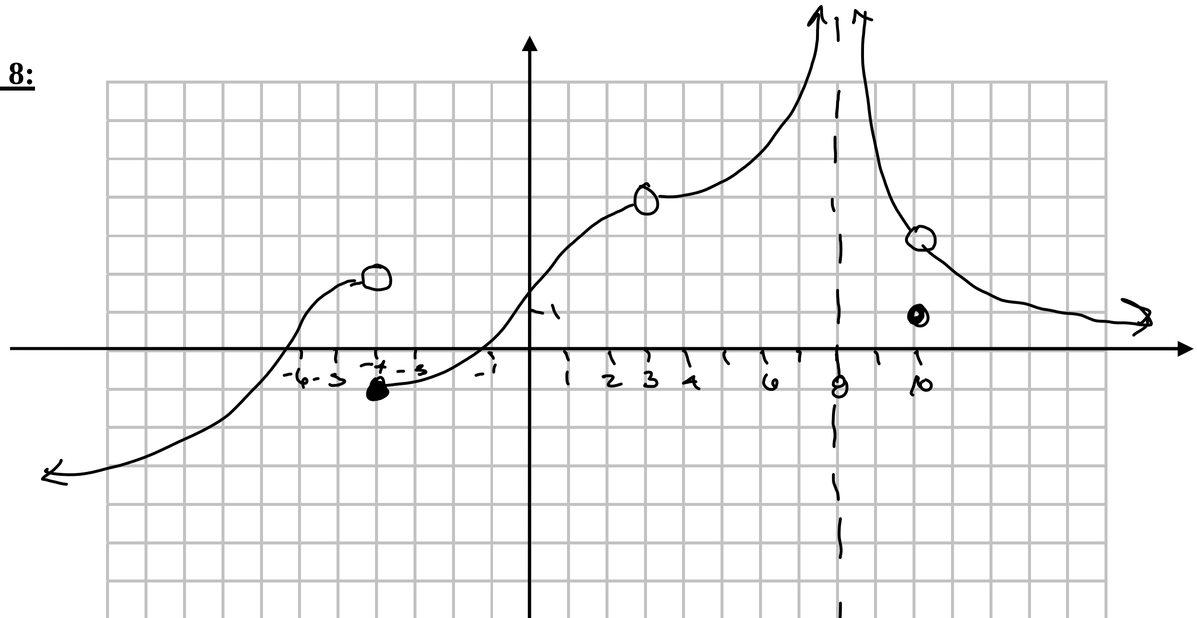


$\boxed{\text{so } \lim\limits_{x\to 1} f(x) \text{ does not exist}}$

Continuity of a function:

In most cases, we can think of a continuous function as one that can be drawn “without lifting your pencil from the paper”. In other words, there are no holes, breaks, or jumps.

Example 8:



This function has discontinuities at $-4, 3, 8, 10$

Domain: $(-\infty, 3) \cup (3, 8) \cup (8, \infty)$

Definition: A function f is continuous at a number a, an interior point of its domain, if

$$\lim_{x \to a} f(x) = f(a)$$

Conditions for Continuity:

In order for f to be continuous at a, all the following conditions must hold:

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

important

At 10: $f(10) \neq \lim_{x \to 10} f(x)$ (both quantities exist)

In Ex 1: which condition is violated at each discontinuity?

At -4: $\lim_{x \to -4} f(x)$ does not exist

At 3: $f(3)$ is not defined

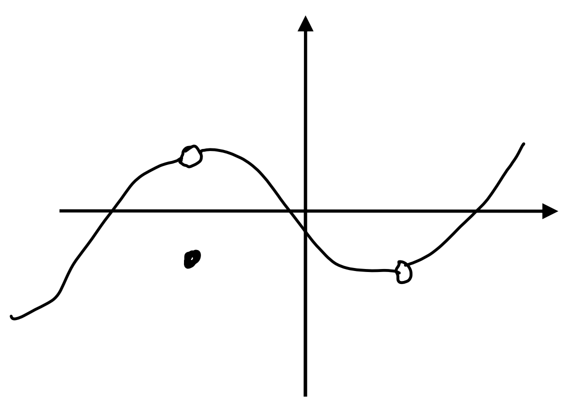
At 8: $f(8)$ does not exist

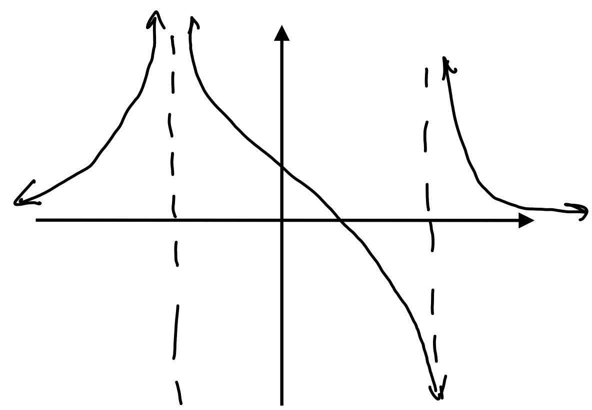
Also, $\lim_{x \to 8} f(x)$ does not exist

Types of discontinuities:

1. Removable discontinuity
2. Infinite discontinuity
3. Jump discontinuity
4. Oscillating discontinuity

What do these look like? For each type of discontinuity, what condition of continuity is violated?





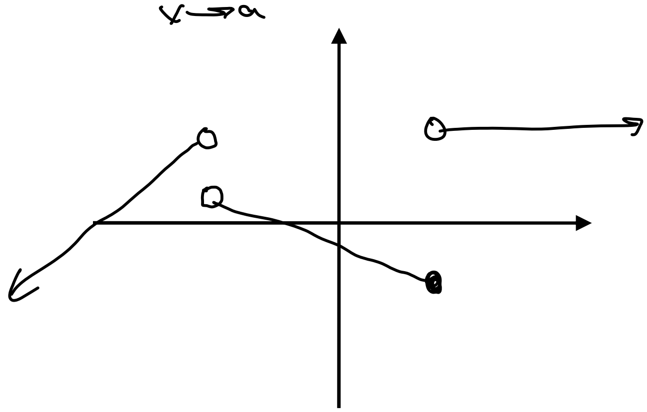
Removable Discontinuities

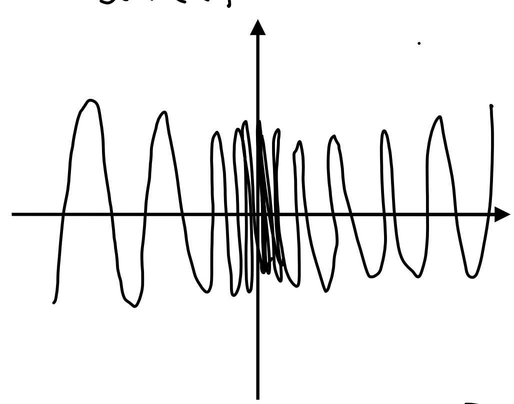
$\lim_{x\to a} f(x)$ exists; and $f(a)$ may or may not exist

If $f(a)$ exists, then $\lim_{x\to a} f(x) \neq f(a)$

Infinite Discontinuities

$\lim_{x\to a} f(x)$ does not exist (either the left- or right-hand limit, or both, is $\pm\infty$)





$f(x) = \sin\left(\frac{1}{x}\right)$

Jump Discontinuities

$\lim_{x\to a} f(x)$ does not exist because $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

Oscillating Discontinuities

($\lim_{x\to a} f(x)$ does not exist)

Continuity at an endpoint of the domain:

A function f is <u>continuous at a number *a*</u>, a left endpoint of its domain, if $\lim_{x\to a^+} f(x) = f(a)$.

A function f is <u>continuous at a number *a*</u>, a right endpoint of its domain, if $\lim_{x\to a^-} f(x) = f(a)$.

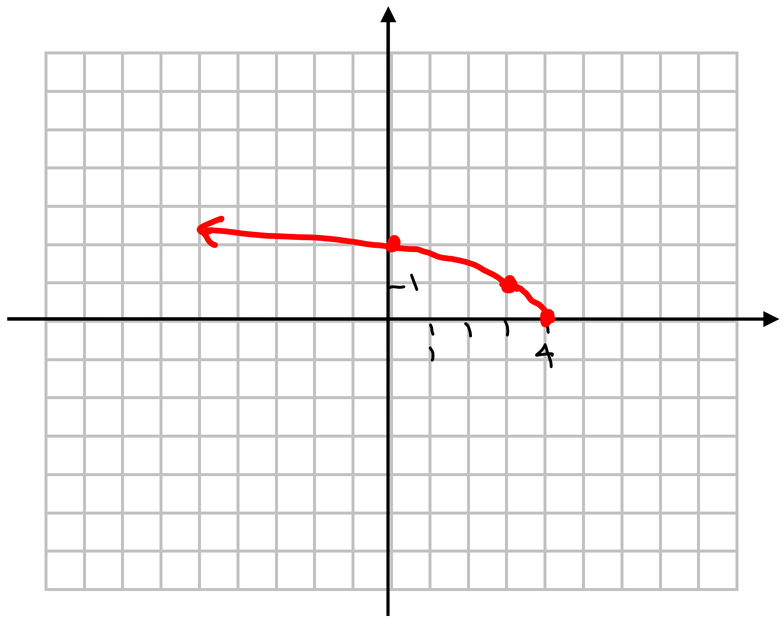
<u>Example 9:</u> $f(x) = \sqrt{4-x}$

Domain: $(-\infty, 4]$

This function is continuous at 4 because

$\lim_{x\to 4^-} f(x) = f(4) = 0$

so f is continuous on $(-\infty, 4]$.



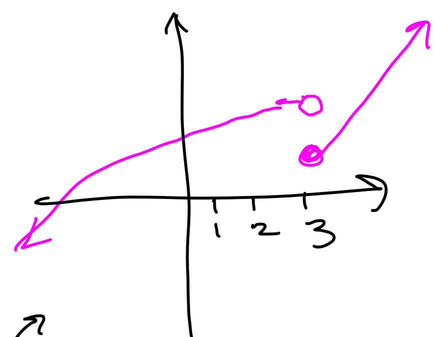
One-sided continuity:

<u>Definition</u>: A function f is continuous from the left at a number a if

$$\lim_{x\to a^-} f(x) = f(a)$$

A function f is continuous from the right at a number a if

$$\lim_{x\to a^+} f(x) = f(a)$$



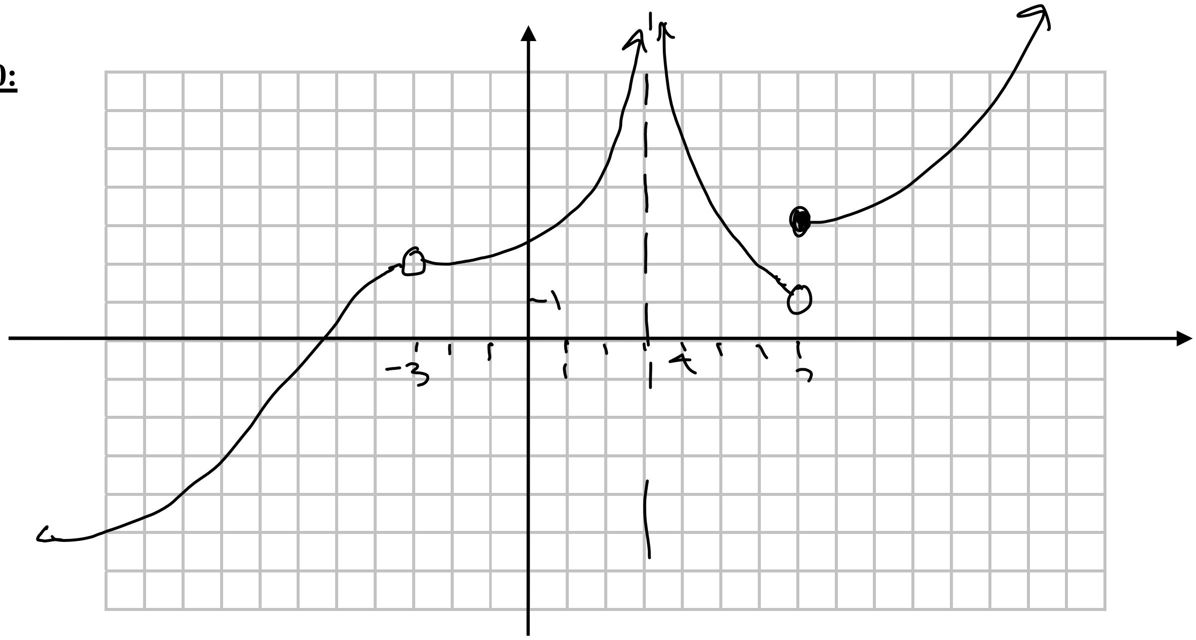
This function is continuous from the right at 3 (because $\lim_{x\to 3^+} = f(3)$)

Continuity on an interval:

<u>Definition</u>: A function f is continuous on an interval if it is continuous at every point in the interval.

(If f is defined only at one side of an endpoint, then only continuity from the left or right is needed for it to be continuous at the endpoint.)

Example 10:



Discontinuities:
-3, 3, 7

On what intervals is the above function continuous?

This function is continuous on

$(-\infty, -3)$, $(3, 7)$, $(7, \infty)$

$(-3, 3)$

Theorem: If f and g are continuous at a and c is a constant, then $f+g$, $f-g$, fg, cf are also continuous at a. The quotient $\dfrac{f}{g}$ is also continuous at a if $g(a) \neq 0$.

Theorem:
Polynomials, rational functions, root functions, and trigonometric functions are continuous at every number in their domains.

<u>Theorem</u>:
If f is continuous at $b=\lim_{x\to a} g(x)$, then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.

<u>Theorem</u>:
If g is continuous at a and f is continuous at $g(a)$, then the composite function $f\circ g$ given by $(f\circ g)(x) = f(g(x))$ is continuous at a.

<u>Example 11:</u> Where is $f(x)=\sin\left(\frac{1}{x}\right)$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Continuous except at 0.
Oscillating discontinuity
Conditions violated at 0: $f(0)$ is not defined; also $\lim_{x\to 0} f(x)$ does not exist.

<u>Example 12:</u> Where is $f(x)=\sqrt{\frac{x^2+1}{(x-1)^3}}$ continuous? At each discontinuity, classify the type of discontinuity and state the condition for continuity that is violated.

Easy to see it's undefined at $x=1$. Where else?
The radicand must be nonnegative.
We need $\frac{x^2+1}{(x-1)^3} \geq 0$

Numerator: always positive for every x
Denominator: will be positive when $x-1>0$
$x>1$

$f(x)$ is defined on $(1,\infty)$

Domain: $(1,\infty)$
Continuous on its domain

<u>Example 13:</u> Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x)=\frac{5x^3-8x^2}{x-7}$$

Domain: Defined for all x except 7.
Domain is $(-\infty,7)\cup(7,\infty)$

Infinite discontinuity at 7.

Conditions for continuity violated: $f(7)$ not defined
$\lim_{x\to 7} f(x)$ does not exist

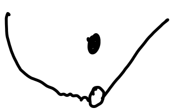
Example 14: Determine the values of *x*, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \frac{x^3 - 3x^2}{x^2 + 4}$$

Denominator is never 0.
(x^2 + positive is always positive)

$\boxed{\text{Domain: } (-\infty, \infty) \quad \text{Continuous everywhere.}}$

Example 15: Determine the values of *x*, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?



$$f(x) = \begin{cases} x+3 & \text{if } x > 0 \\ 4 & \text{if } x = 0 \\ x^2+3 & \text{if } x < 0 \end{cases}.$$

Any discontinuities within any of the "pieces"? No.
We then look at boundary between pieces.
Does $\lim_{x\to 0} f(x)$ exist?

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(x^2+3) = 0^2+3 = 3$; $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+3) = 0+3 = 3$

Thus $\lim_{x\to 0} f(x) = 3$. But $f(0) = 4$. Since $f(0) \neq \lim_{x\to 0} f(x)$, f is $\boxed{\text{not continuous at } 0.}$ (removable discontinuity)

Example 16: Determine the values of *x*, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated. At which of these numbers is it continuous from the left or right?

$$f(x) = \begin{cases} 5x & \text{if } x > 1 \\ 5 & \text{if } x = 1 \\ x+5 & \text{if } x < 1 \end{cases}.$$

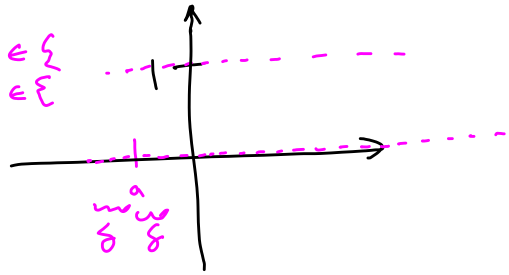
$\lim_{x\to 1^-}$

Example 17: Determine the values of x, if any, at which the function is discontinuous.

$$g(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

Domain: $(-\infty, \infty)$ all real numbers.

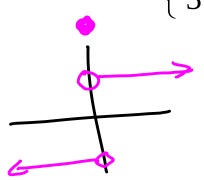
Discontinuous everywhere



Example 18: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 3 & x = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{|x|}{x} = \dfrac{x}{x} = 1 & x > 0 \\ \dfrac{|x|}{x} = \dfrac{-x}{x} = -1 & x < 0 \\ 3 & x = 0 \end{cases}$$



(not to scale)

Jump discontinuity

$\lim_{x \to 0} f(x)$ does not exist

Example 19: Determine the values of x, if any, at which the function is discontinuous. For each, classify the type of discontinuity and state the condition for continuity that is violated.

$$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & x \neq 5 \\ 10 & x = 5 \end{cases}$$

Note: For $x \neq 5$, $f(x) = \dfrac{(x+5)(x-5)}{x-5}$

("cancelled" version would be $g(x) = x+5$.)

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \frac{(x+5)(x-5)}{(x-5)} = \lim_{x \to 5^-} (x+5) = 5+5 = 10$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \frac{(x+5)(x-5)}{x-5} = \lim_{x \to 5^+} (x+5) = 5+5 = 10.$$

So, $\lim_{x \to 5} f(x) = 10$. Also, $f(5) = 10$. Since these are equal, f is continuous at 5. No other potential discontinuities to analyze, so f is continuous on $(-\infty, \infty)$

Example 20: Find a function g that agrees with f for $x \neq 1$ and is continuous on $(-\infty, \infty)$.

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

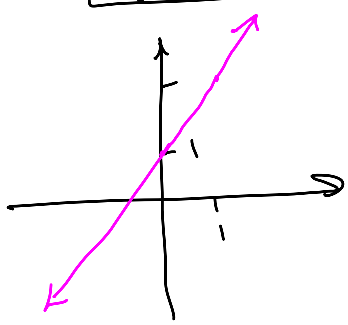
Domain of f: everywhere except $x \neq 1$

Domain: $(-\infty, 1) \cup (1, \infty)$

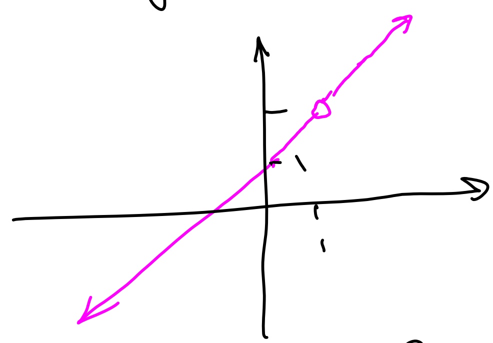
$$f(x) = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

Final answer → $\boxed{g(x) = x+1}$

Domain of g: $(-\infty, \infty)$



graph of g

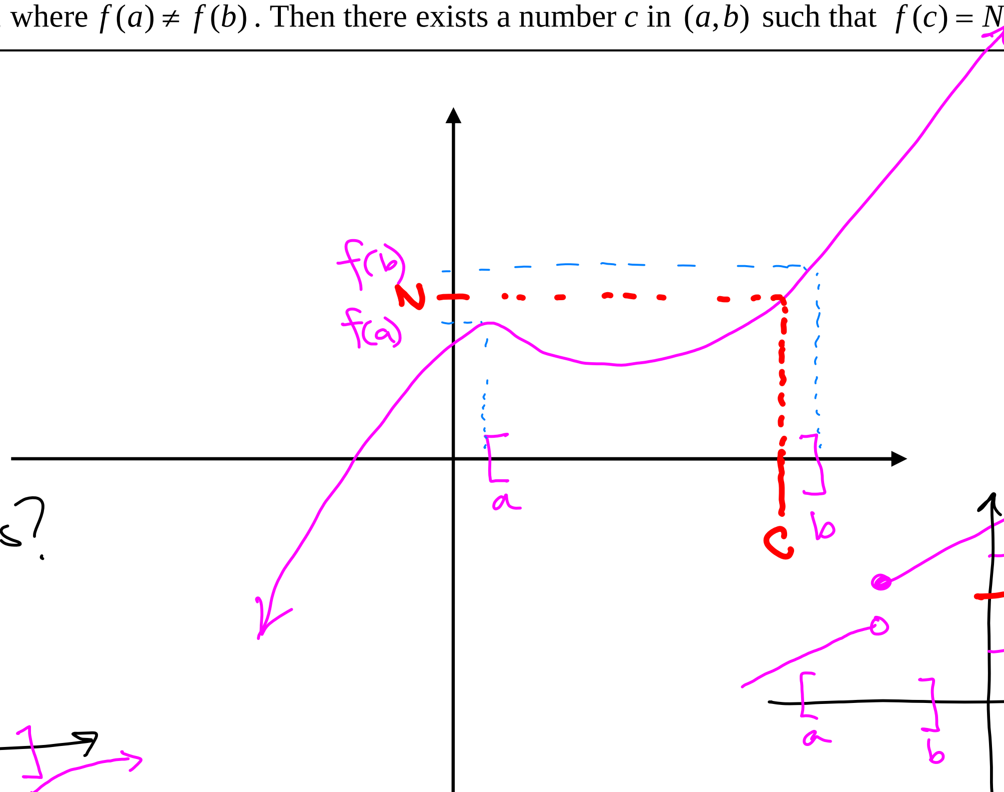


graph of f

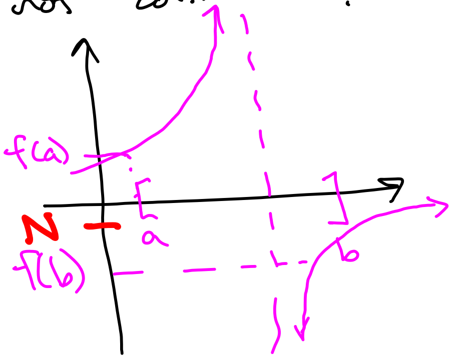
Note: $g(1) = 1+1 = 2$

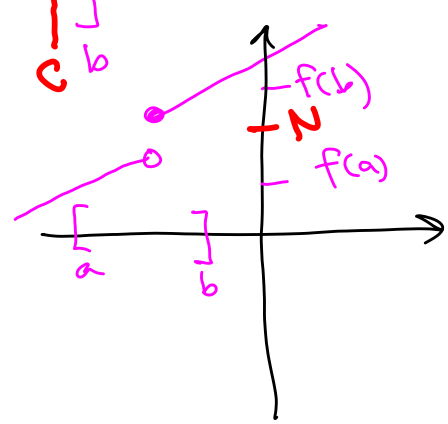
The Intermediate Value Theorem:

Suppose that *f* is continuous on the closed interval $[a,b]$ and let *N* be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number *c* in (a,b) such that $f(c) = N$.



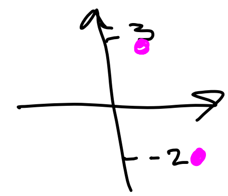
What if f is not continuous?





Example 21: Show that $f(x) = x^3 - 4x^2 + 6$ has a zero between 1 and 2.

f is continuous on $(-\infty, \infty)$. So certainly continuous on $[1, 2]$.



$f(1) = 1^3 - 4(1)^2 + 6 = 1 - 4 + 6 = 3$

$f(2) = 2^3 - 4(2)^2 + 6 = 8 - 16 + 6 = -2$

Because $-2 < 0 < 3$, there must be a c in the interval $(1, 2)$ such that $f(c) = 0$. (from Intermediate Value theorem)

Example 22: Is there a number that is equal to its own cosine?

In other words, does $x = \cos x$ have a solution?

Rewrite: $\cos x - x = 0$

Let $f(x) = \cos x - x$. Does this have a zero?

Think of an x-value that makes $f(x)$ positive; also an x-value that makes $f(x)$ negative.

Try $x = \frac{\pi}{2}$: $f(\frac{\pi}{2}) = \cos\frac{\pi}{2} - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$ neg.

Try $x = 0$: $f(0) = \cos 0 - 0 = 1 - 0 = 1$ pos.

So, from Int. Value. Thm, because f is continuous, there must be a c in $(0, \frac{\pi}{2})$ such that $f(c) = 0$.