1.5: Infinite Limits

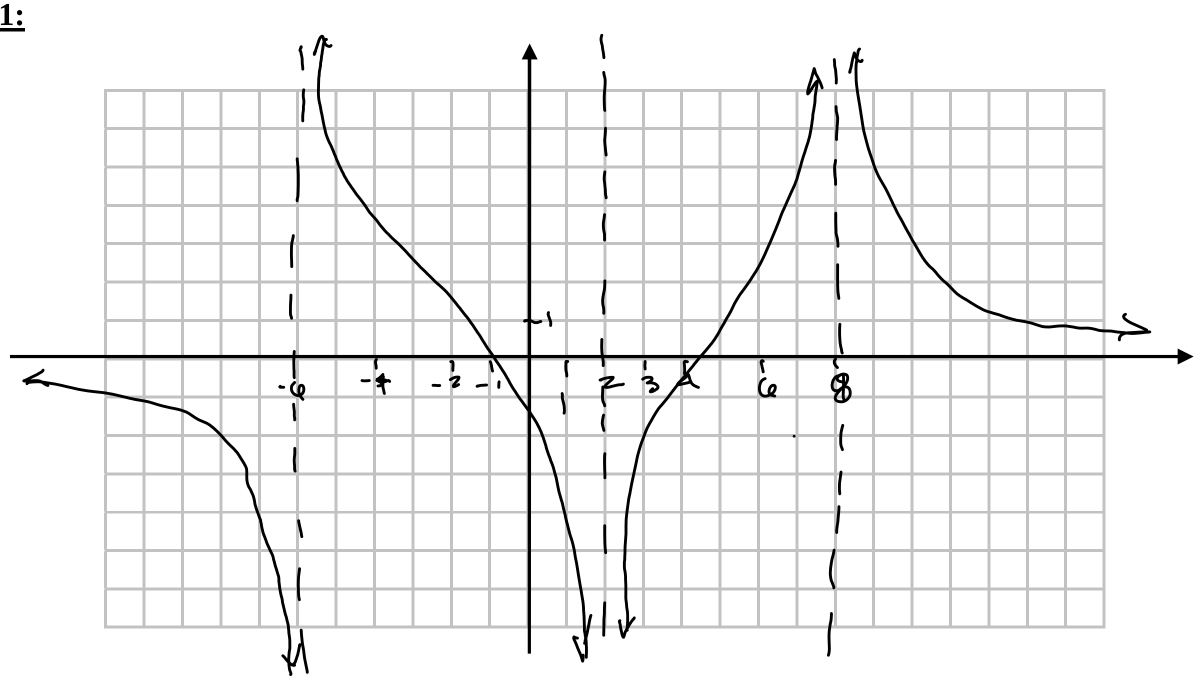
There are two types of limits involving infinity. $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

Limits at infinity, written in the form $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, are related to horizontal asymptotes and will be covered in Section 3.5, as we learn to graph functions.

Infinite limits take the form of statements like $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. Infinite limits can result in vertical asymptotes, also important in graphing functions.

Determining infinite limits from a graph:

Example 1:



$\lim_{x\to -6} f(x)$ does not exist

$\lim_{x\to -6^-} f(x) = -\infty$

$\lim_{x\to -6^+} f(x) = \infty$

these also do not exist

$\lim_{x\to 2} f(x) = -\infty$ (does not exist)

$\lim_{x\to 8} f(x) = \infty$ (does not exist)

Preview of 3.5:

$\lim_{x\to \infty} f(x) = 0$

$\lim_{x\to -\infty} f(x) = 0$

Determining infinite limits from a table of values:

__**Example 2:**__ Use a table of values to determine $\lim_{x\to 2} f(x) = \dfrac{x+6}{x-2}$.

Note: f is undefined at 2.

x	$f(x)=\frac{x+6}{x-2}$
1.9	-79
1.95	-159
1.99	-799
1.999	-7999
1.9999	-79999

$\downarrow$

$-\infty$

Also approach from the right (from above 2)

x	$f(x)=\frac{x+6}{x-2}$
2.05	161
2.01	801
2.001	8001
2.0001	80001
2.00001	800001
2.000001	8000001

$\downarrow$

∞

From table,

$\lim_{x\to 2^-} f(x) = -\infty$ (does not exist)

$\lim_{x\to 2^+} f(x) = \infty$ (does not exist)

$\lim_{x\to 2} f(x)$ does not exist

__**Example 3:**__ Use a table of values to determine $\lim_{x\to 2} \cancel{f(x) = \dfrac{x+6}{(x-2)^2}}$.

$\lim_{x\to 2} \left(\dfrac{x+6}{(x-2)^2}\right)$.

Denominator always + tiny

Numerator ≈ 8

$\dfrac{8}{+\text{tiny}} \to +\text{huge}$. I conclude that

$$\boxed{\lim_{x\to 2} \frac{x+6}{(x-2)^2} = \infty.}$$

(confirmed w/table)

> __Important:__
>
> Statements such as $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$, or $\lim_{x\to a^+} f(x) = -\infty$ do NOT mean the limit exists. Rather, these statements mean that the limit DOES NOT EXIST. and they describe the reason that the limit fails to exist (by describing the behavior of the function near the given x-value).

Formal definition of an infinite limit:

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that for every positive real number M, there exists a number $\delta > 0$ such that

$$f(x) > M \text{ whenever } 0 < |x-a| < \delta .$$

Similarly, $\lim_{x\to a} f(x) = \overset{-\infty}{\cancel{\infty}}$ means that for every negative real number N, there exists a number $\delta > 0$ such that

$$f(x) < N \text{ whenever } 0 < |x-a| < \delta .$$

Example 4: Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

As $x \to 0^-$, $\frac{1}{x^2} \to \frac{1}{(-\text{tiny})^2} \to \frac{1}{+\text{tiny}} \to +\text{huge}$

Similar with $x \to 0^+$

Scratchwork

Let $M > 0$. (Choose any $M>0$)

Want to show that $\frac{1}{x^2} > M$

$1 > Mx^2$

$\frac{1}{M} > x^2$

$x^2 < \frac{1}{M}$

$\sqrt{x^2} < \sqrt{\frac{1}{M}}$

$|x| < \frac{\sqrt{1}}{\sqrt{M}}$

$|x| < \frac{1}{\sqrt{M}}$

$|x-0| < \frac{1}{\sqrt{M}}$

Choose $\delta = \frac{1}{\sqrt{M}}$

Proof: Let $M>0$. (Let M be any positive number.)

Let $\delta = \frac{1}{\sqrt{M}}$. Suppose $0 < |x-0| < \frac{1}{\sqrt{M}}$.

then $0 < |x| < \frac{1}{\sqrt{M}}$ and so $-\frac{1}{\sqrt{M}} < x < \frac{1}{\sqrt{M}}$

Because $x < \frac{1}{\sqrt{M}}$, we have $x\sqrt{M} < 1$ see next page

Let's assume (temporarily) that $x > 0$.

$$x\sqrt{m} < 1$$

$$\sqrt{m} < \frac{1}{x}$$

$$m < \frac{1}{x^2}$$

Now let's assume that $x < 0$.

$$x\sqrt{m} < 1$$

$$\sqrt{m} > \frac{1}{x}$$ (dividing by a neg number, so we reverse the sign.

$\frac{1}{x}$ is neg, $\sqrt{m}$ is positive

$$\frac{1}{x} < \sqrt{m}$$

$$|x| < \frac{1}{\sqrt{m}}$$

$$\frac{1}{|x|} > \frac{\sqrt{m}}{1}$$

Then square both sides:

$$\left(\frac{1}{|x|}\right)^2 > \left(\sqrt{m}\right)^2$$

$$\frac{1}{x^2} > m$$

Evaluating infinite limits from an equation: Try 3.99: $\frac{3.99-8}{3.99-4}$

Example 5: Determine $\lim\limits_{x\to 4}\frac{x-8}{x-4}$.

As $x \to 4^-$, $\frac{x-8}{x-4} \to \frac{-4}{-\text{tiny}} \to +\text{huge}$ so $\lim\limits_{x\to 4^-}\frac{x-8}{x-4} = +\infty$

As $x \to 4^+$, $\frac{x-8}{x-4} \to \frac{-4}{+\text{tiny}} \to -\text{huge}$, so $\lim\limits_{x\to 4^+}\frac{x-8}{x-4} = -\infty$

Try $\frac{4.01-8}{4.01-4}$

$\boxed{\text{So } \lim\limits_{x\to 4}\frac{x-8}{x-4} \text{ does not exist.}}$

Example 6: Determine $\lim\limits_{x\to 2^-}\frac{x-8}{x^2-4}$.

As $x \to 2^-$, $\frac{x-8}{x^2-4} \to \frac{-6}{-\text{tiny}} \to +\text{huge}$

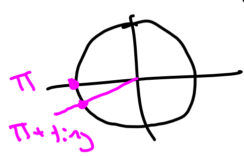
Try: 1.99: $\frac{1.99-8}{(1.99)^2-4}$

$\boxed{\text{So } \lim\limits_{x\to 2^-}\frac{x-8}{x^2-4} = +\infty}$

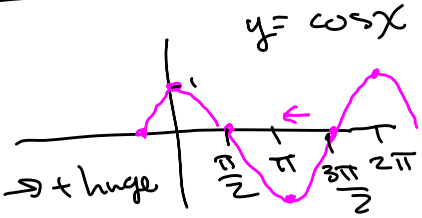
Example 7: Determine $\lim\limits_{x\to 3}\frac{x^3-2}{(x-3)^2}$.

$\boxed{\lim\limits_{x\to 3}\frac{x^3-2}{(x-3)^2} = \infty}$

Example 8: Determine $\lim\limits_{x\to \pi^+}\frac{\sin\left(\frac{x}{3}\right)}{1+\cos x}$.



y = cos x



Try 3.15: $\frac{\sin\left(\frac{3.15}{3}\right)}{1+\cos 3.15}$

As $x \to \pi^+$, $\frac{\sin\frac{x}{3}}{1+\cos x} \to \frac{\sin(\pi/3)}{1-0.99} \to \frac{\sqrt{3}/2}{+\text{tiny}} \to +\text{huge}$

$\lim\limits_{x\to\pi^+}\frac{\sin(x/3)}{1+\cos x} = +\infty$

Example 9: Determine $\lim\limits_{x\to \pi/6}\tan(3x)$.

Note: $\tan\left(3\left(\frac{\pi}{6}\right)\right) = \tan\left(\frac{\pi}{2}\right) = \frac{\sin(\pi/2)}{\cos(\pi/2)}$ is undefined.

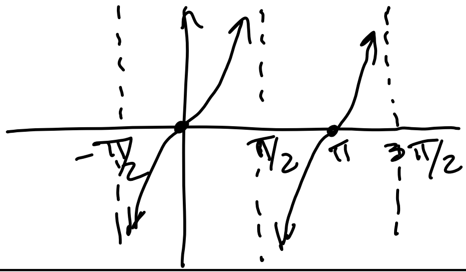
as $x \to \frac{\pi}{6}^+$, $\tan 3x = \frac{\sin(3x)}{\cos(3x)} \to \frac{\sin(3\pi/6)}{-\text{tiny}} \to \frac{1}{-\text{tiny}} \to -\text{huge}$

As $x \to \frac{\pi}{6}^-$, $\tan 3x = \frac{\sin 3x}{\cos 3x} \to \frac{1}{+\text{tiny}} \to +\text{huge}$

$\boxed{\lim\limits_{x\to\pi/6}\tan(3x) \text{ does not exist.}}$

See next page

Note: graph of $y = \tan x$



Vertical asymptotes:

Vertical Asymptotes:

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

Example 10: Determine the asymptotes of $f(x) = \dfrac{x-2}{x+3}$. Sketch the graph.

Horizontal Asymptote: $y = \frac{1}{1}$

$y = 1$

Intercepts:

$x = 0 \Rightarrow y = \frac{0-2}{0+3} = -\frac{2}{3}$

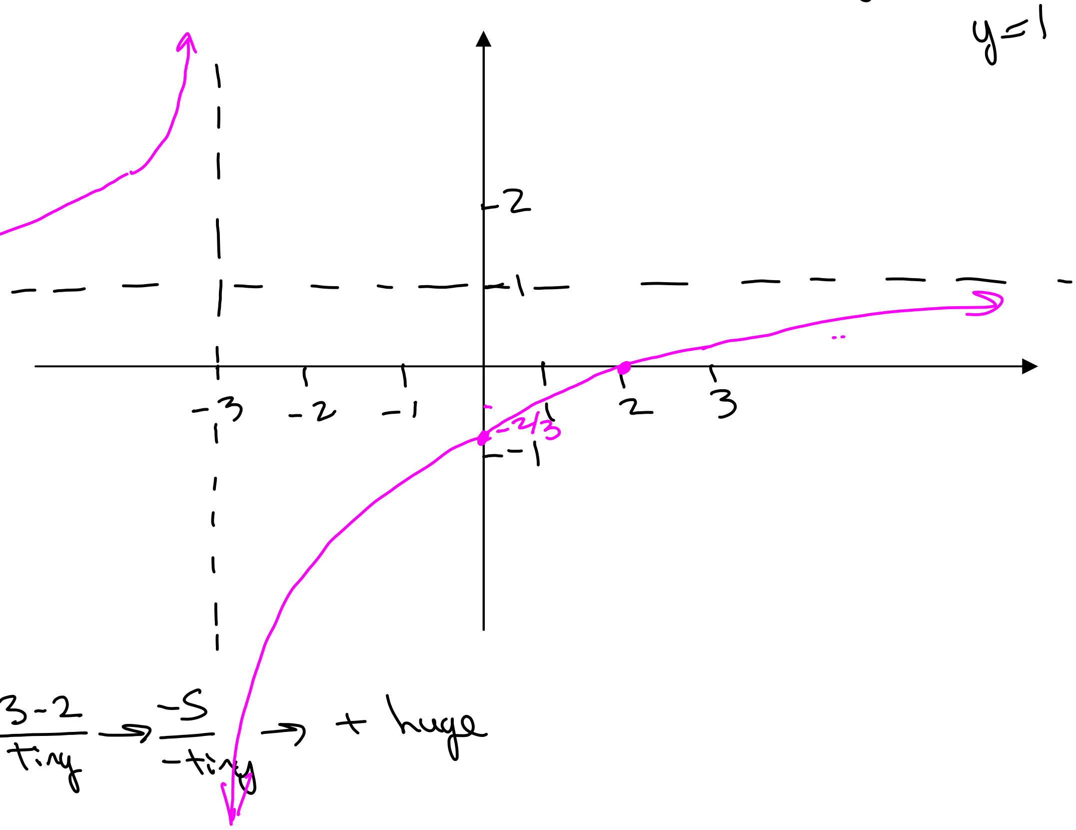
$y = 0 \Rightarrow 0 = \frac{x-2}{x+3}$

$0 = x - 2$

$x = 2$

y-int: $-\frac{2}{3}$

x-int: 2



Vertical Asymptote:

$x = -3$

$f(x) = \frac{x-2}{x+3}$

As $x \to -3^-$, $y \to \frac{-3-2}{-\text{tiny}} \to \frac{-5}{-\text{tiny}} \to +$ huge

Try -3.01: $\frac{-3.01-2}{-3.01+3}$

As $x \to -3^+$, $y \to \frac{-5}{+\text{tiny}} \to -$ huge

Try -2.99: $\frac{-2.99-2}{-2.99+3}$

Example 11: Determine the vertical asymptotes of $f(x)=\dfrac{3}{x^2-1}$. Sketch the graph.

$f(x)=\dfrac{3}{x^2-1}=\dfrac{3}{(x+1)(x-1)}$

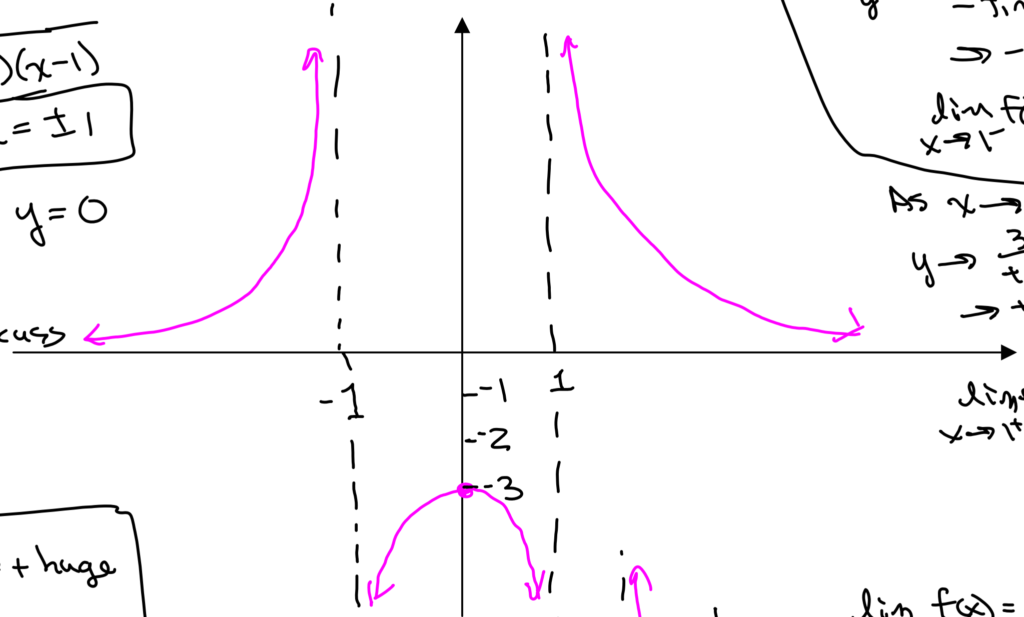
Vertical Asymptotes: $\boxed{x=\pm 1}$

Horizontal Asymptote: $y=0$

(using College Algebra knowledge... we'll discuss more in 3.5)

x-intercept: none

y-intercept: -3



As $x\to 1^-$, $y\to \dfrac{3}{-\text{tiny}}\to -\text{huge}$

$\lim\limits_{x\to 1^-} f(x)=-\infty$

As $x\to 1^+$, $y\to \dfrac{3}{+\text{tiny}}\to +\text{huge}$

$\lim\limits_{x\to 1^+} f(x)=+\infty$

As $x\to -1^-$, $y\to \dfrac{3}{+\text{tiny}}\to +\text{huge}$

$\dfrac{3}{(-1.01)^2-1}$ $\lim\limits_{x\to -1^-} f(x)=+\infty$

As $x\to -1^+$, $y\to \dfrac{3}{-\text{tiny}}\to -\text{huge}$. $\lim\limits_{x\to -1^+} f(x)=-\infty$

Example 12: Determine the vertical asymptotes of $f(x)=\dfrac{x^2-9}{x^2-5x+6}$. Sketch the graph.

$f(x)=\dfrac{(x+3)(x-3)}{(x-3)(x-2)}$

except at $x=3$, this function is identical to

$g(x)=\dfrac{x+3}{x-2}$

("cancelled" version of f)

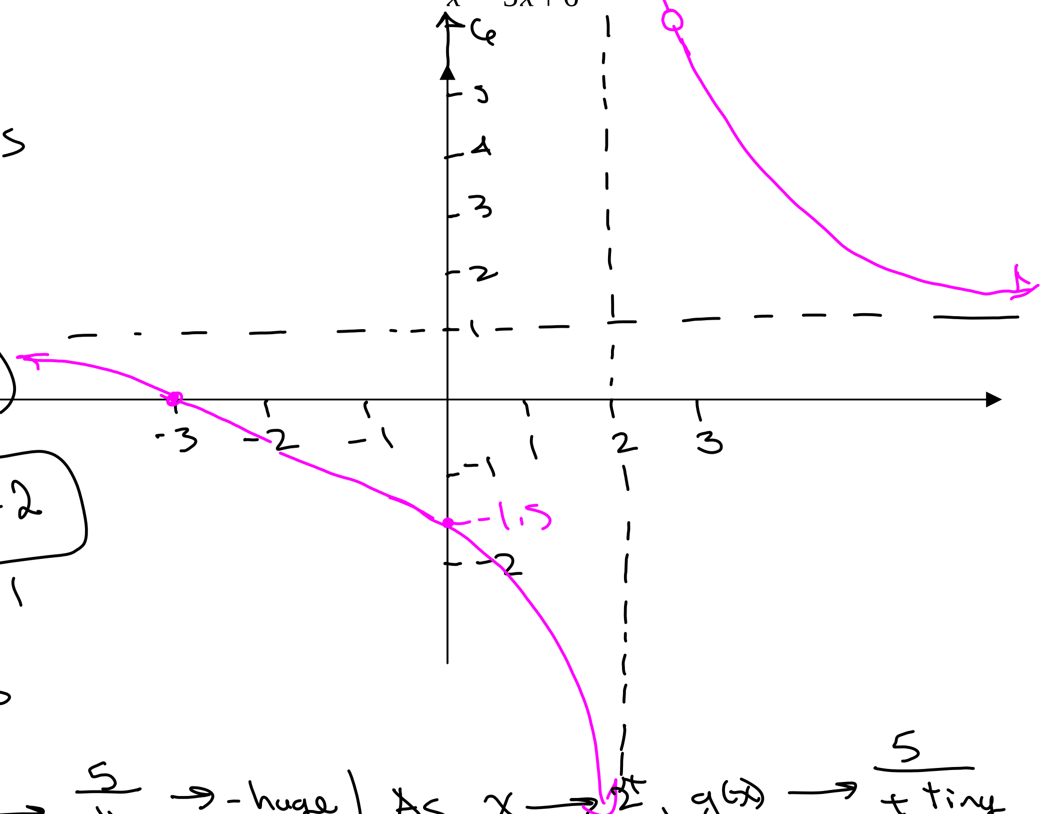
Graph g 1st:

$\boxed{\text{Vertical asymptote: } x=2}$

Horizontal asymptote: $y=1$

x-intercept: -3

y-intercept: $-\frac{3}{2}=-1.5$



As $x\to 2^-$, $g(x)\to \dfrac{5}{-\text{tiny}}\to -\text{huge}$

$\dfrac{1.99+3}{1.99-2}$

$\lim\limits_{x\to 2^-} g(x)=-\infty$

As $x\to 2^+$, $g(x)\to \dfrac{5}{+\text{tiny}}\to +\text{huge}$

$\dfrac{2.01+3}{2.01-2}$

$\lim\limits_{x\to 2^+} g(x)=+\infty$

See next page

Ex 12 cont'd:

Once you graph g, we modify it to get the graph of f. Function f has a removable discontinuity ("hole") at $x=3$.

Find the y-coordinate for the removable discontinuity:

Go back to $g(x) = \frac{x+3}{x-2}$.

$$g(3) = \frac{3+3}{3-2} = \frac{6}{1} = 6$$

Use your eraser to make a removable discontinuity at $(3,6)$.