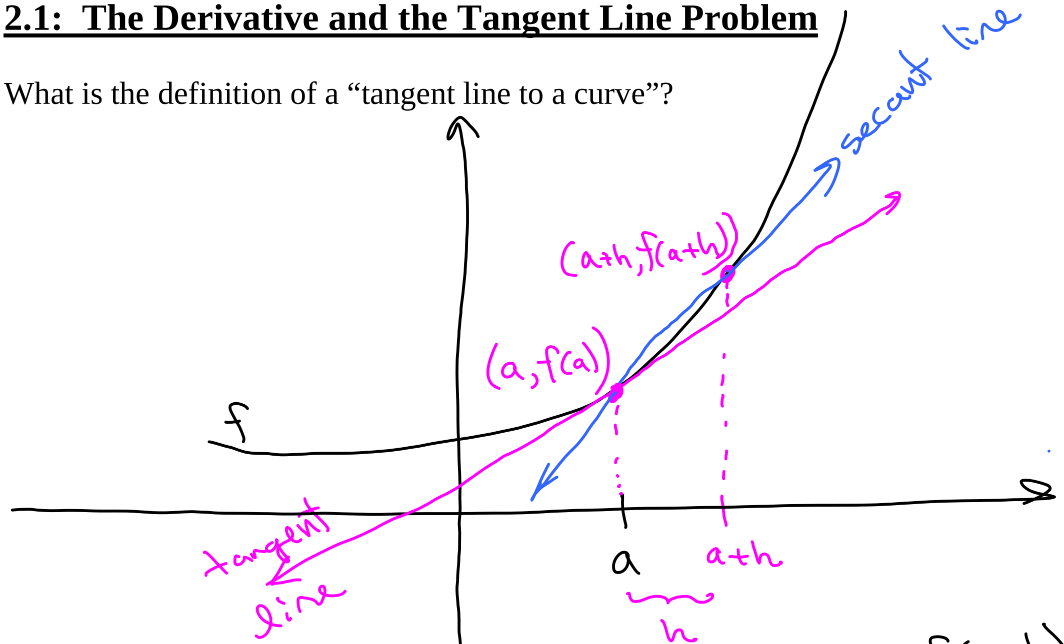
2.1: The Derivative and the Tangent Line Problem

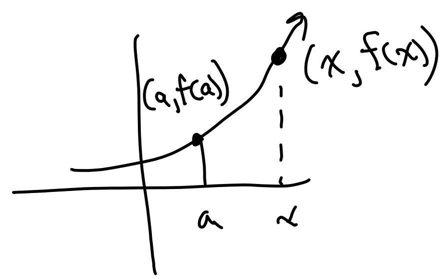
What is the definition of a "tangent line to a curve"?



slope of secant line $= \dfrac{f(a+h)-f(a)}{h}$

To answer the difficulty in writing a clear definition of a tangent line, we can define it as the limiting position of the secant line as the second point approaches the first.

<u>Definition</u>: The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through *P* with slope



$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \text{ provided this limit exists.}$$

Equivalently,

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ provided this limit exists.}$$

<u>Note</u>: If the tangent line is vertical, this limit does not exist. In the case of a vertical tangent, the equation of the tangent line is $x = a$.

<u>Note</u>: The slope of the tangent line to the graph of *f* at the point $(a, f(a))$ is also called the <u>slope of the graph</u> of *f* at $x = a$.

<u>How to get the second expression for slope</u>: Instead of using the points $(a, f(a))$ and $(x, f(x))$ on the secant line and letting $x \to a$, we can use $(a, f(a))$ and $(a+h, f(a+h))$ and let $h \to 0$.

Find eqn of tangent line: $y - y_1 = m(x - x_1)$ OR $y = mx + b$

$m = 24,\ x_1 = 3,\ y_1 = 37 \Rightarrow y - 37 = 24(x-3)$

$y - 37 = 24x - 72$

$\boxed{y = 24x - 35}$

$37 = 24(3) + b$

$b = -35$

$\boxed{y = 24x - 35}$

Example 1: Find the slope of the curve $y = 4x^2 + 1$ at the point $(3, 37)$. Find the equation of the tangent line at this point.

Define $\boxed{f(x) = 4x^2 + 1}$

Slope: $m = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{\overbrace{[4(3+h)^2 + 1]}^{f(3+h)} - \overbrace{[4(3)^2 + 1]}^{f(3)}}{h}$

$= \lim_{h \to 0} \frac{4(9 + 6h + h^2) + 1 - 37}{h}$

$= \lim_{h \to 0} \frac{\cancel{36} + 24h + 4h^2 - \cancel{36}}{h} = \lim_{h \to 0} \frac{24h + 4h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(24 + 4h)}{\cancel{h}}$

$= \lim_{h \to 0} (24 + 4h) = 24 + 4(0) = \boxed{24}$ slope at $(3, 37)$. still need eqn of tangent line ... see above

Example 2: Find an equation of the tangent line to the curve $y = x^3$ at the point $(1,1)$.

Alternative Def'n:

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

we'll use $a = 1$

Slope $= m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^3 - 1^3}{x - 1} = \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + 1x + 1^2)}{\cancel{x - 1}} = \lim_{x \to 1} (x^2 + x + 1)$

$= 1^2 + 1 + 1 = 3 = \text{slope}$

Could do polynomial long division, or remember formula for factoring $a^3 - b^3$:

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$y = mx + b$

$1 = 3(1) + b$

$-2 = b$

Thus $\boxed{y = 3x - 2}$

OR

$y - y_1 = m(x - x_1)$

$y - 1 = 3(x - 1)$

$y - 1 = 3x - 3$

$\boxed{y = 3x - 2}$

Example 3: Determine the equation of the tangent line to $f(x) = \sqrt{x}$ at the point where $x = 2$.

The derivative:

The derivative of a function at x is the slope of the tangent line at the point $(x, f(x))$. It is also the instantaneous rate of change of the function at x.

f' "f prime"

> Definition: The *derivative* of a function f at x is the function f' whose value at x is given by
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ provided this limit exists.}$$

The process of finding derivatives is called <u>differentiation</u>. To <u>differentiate</u> a function means to find its derivative.

Equivalent ways of defining the derivative:

Result in a function of x

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(Our book uses this one. It is identical to the definition above, except uses Δx in place of h.)

$$f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}$$

Result in a number

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(Gives the derivative at the specific point where $x = a$.)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(Gives the derivative at the specific point where $x = a$.)

<u>Example 4:</u> Suppose that $g(x) = \dfrac{x^2 - 6x}{3}$. Determine $g'(x)$ and $g'(3)$.

$$g(x) = \frac{1}{3}(x^2 - 6x)$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{3}\left[(x+h)^2 - 6(x+h)\right] - \frac{1}{3}(x^2 - 6x)}{h}$$

$$= \frac{1}{3} \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{6x} - 6h - \cancel{x^2} + \cancel{6x}}{h}$$

$$= \frac{1}{3} \lim_{h \to 0} \frac{2xh + h^2 - 6h}{h} = \frac{1}{3} \lim_{h \to 0} \frac{\cancel{h}(2x + h - 6)}{\cancel{h}}$$

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$$= \frac{1}{3} \lim_{h \to 0} (2x + h - 6) = \frac{1}{3}(2x + 0 - 6) = \boxed{\frac{2}{3}x - 2 = g'(x)}$$

$$g'(3) = \frac{2}{3}(3) - 2 = 2 - 2 = \boxed{0} \quad \text{slope of tangent line at } x = 3$$

Example 5: Suppose that $f(x) = \sqrt{x^2+1}$. Find the equation of the tangent line at the point where $x = 2$.

Find derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x^2+2xh+h^2+1} - \sqrt{x^2+1}}{h}\left(\frac{\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}}{\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}}\right)$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2+1 - (x^2+1)}{h\left(\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}\right)}$$

using $(a-b)(a+b) = a^2 - b^2$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h\left(\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}\right)} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}\left(\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0} \frac{2x+h}{\sqrt{x^2+2xh+h^2+1} + \sqrt{x^2+1}} = \frac{2x+0}{\sqrt{x^2+0+0+1} + \sqrt{x^2+1}} = \frac{2x}{2\sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}}. \quad \text{Then slope is } f'(2) = \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{5}}$$

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Example 6: Determine the equation of the tangent line to $f(x) = \dfrac{x-2}{x^2+1}$ at the point $\left(-2, -\dfrac{4}{5}\right)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h-2}{(x+h)^2+1} - \frac{x-2}{x^2+1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h-2}{x^2+2xh+h^2+1} - \frac{x-2}{x^2+1}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x+h-2}{x^2+2xh+h^2+1}\left(\frac{x^2+1}{x^2+1}\right) - \frac{x-2}{x^2+1}\left(\frac{x^2+2xh+h^2+1}{x^2+2xh+h^2+1}\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{x^3} + \cancel{x} + hx^2 + h - \cancel{2x^2} - \cancel{2} - (\cancel{x^3} + 2x^2h + xh^2 + \cancel{x} - \cancel{2x^2} - 4xh - 2h^2 - \cancel{2})}{(x^2+1)(x^2+2xh+h^2+1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{hx^2 + h - 2x^2h - xh^2 + 4xh + 2h^2}{(x^2+1)(x^2+2xh+h^2+1)}\right]$$

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$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{h}(x^2+1-2x^2-xh+4x+2h)}{(x^2+1)(x^2+2xh+h^2+1)}\right] = \lim_{h\to 0}\left[\frac{-x^2+1+4x+2h}{(x^2+1)(x^2+2xh+h^2+1)}\right]$$

Ex 5 cont'd) Find equation of tangent line at point where $x=2$.

Let's use $y-y_1 = m(x-x_1)$:

$f(x) = \sqrt{x^2+1}$, so $y_1 = f(2) = \sqrt{2^2+1} = \sqrt{5}$

[use $x_1 = 2$, $y_1 = \sqrt{5}$, $m = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$]

$$y - y_1 = m(x-x_1)$$
$$y - \sqrt{5} = \frac{2\sqrt{5}}{5}(x-2)$$
$$y - \sqrt{5} = \frac{2\sqrt{5}}{5}x - \frac{4\sqrt{5}}{5}$$

$$y = \frac{2\sqrt{5}}{5}x - \frac{4\sqrt{5}}{5} + \sqrt{5}$$
$$y = \frac{2\sqrt{5}}{5}x - \frac{4\sqrt{5}}{5} + \frac{5\sqrt{5}}{5}$$
$$\boxed{y = \frac{2\sqrt{5}}{5}x + \frac{\sqrt{5}}{5}}$$

Ex 6: Continued

$$\lim_{h \to 0}\left[\frac{-x^2+1+4x+2h}{(x^2+1)(x^2+2xh+h^2+1)}\right] = \frac{-x^2+1+4x+2(0)}{(x^2+1)(x^2+2x(0)+0^2+1)}$$

$$= \frac{-x^2+1+4x}{(x^2+1)(x^2+1)} = \frac{-x^2+1+4x}{(x^2+1)^2} = f'(x)$$

Find slope:

$$f'(-2) = \frac{-(-2)^2+1+4(-2)}{((-2)^2+1)^2} = \frac{-4+1-8}{5^2} = -\frac{11}{25}$$

Then find eqn of tangent line.

Summary:

The slope of the secant line between two points is often called a difference quotient. The difference quotient of *f* at *a* can be written in either of the forms below.

$$\frac{f(x)-f(a)}{x-a} \qquad\qquad \frac{f(a+h)-f(a)}{h}.$$

Both of these give the slope of the secant line between two points: $(x, f(x))$ and $(a, f(a))$ or, alternatively, $(a, f(a))$ and $(a+h, f(a+h))$.
The slope of the secant line is also the average rate of change of *f* between the two points.

Test question!

The derivative of *f* at *a* is:

1) the limit of the slopes of the secant lines as the second point approaches the point $(a, f(a))$.

2) the slope of the tangent line to the curve $y = f(x)$ at the point where $x = a$.

3) the (instantaneous) rate of change of f with respect to *x* at *a*.

4) $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ (limit of the difference quotient)

5) $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ (limit of the difference quotient)

Common notations for the derivative of $y = f(x)$:

$f'(x) \qquad \dfrac{d}{dx}f(x) \qquad y' \qquad D_x f(x) \qquad \dfrac{dy}{dx} \qquad Df(x)$

The notation $\dfrac{dy}{dx}$ was created by Gottfried Wilhelm Leibniz and means $\dfrac{dy}{dx} = \lim\limits_{\Delta x\to 0}\dfrac{\Delta x}{\Delta y}$.
To evaluate the derivative at a particular number *a*, we write

$$f'(a) \text{ or } \left.\frac{dy}{dx}\right|_{x=a}$$

Differentiability: "Differentiate" means to take the derivative.

Definition: A function f is *differentiable* at a if $f'(a)$ exists. It is *differentiable on an open interval* if it is differentiable at every number in the interval.

$$\boxed{\text{Theorem: If } f \text{ is differentiable at } a\text{, then } f \text{ is continuous at } a.}$$

Note: The converse is not true—there are functions that are continuous at a number but not differentiable.

Note: Open intervals: (a,b), $(-\infty,a)$, (a,∞), $(-\infty,\infty)$.

Closed intervals: $[a,b]$, $(-\infty,a]$, $[a,\infty)$, $(-\infty,\infty)$.

To discuss differentiability on a closed interval, we need the concept of a *one-sided derivative*.

Derivative from the left: $\lim\limits_{x\to a^-}\dfrac{f(x)-f(a)}{x-a}$

Derivative from the right: $\lim\limits_{x\to a^+}\dfrac{f(x)-f(a)}{x-a}$

For a function f to be differentiable on the closed interval $[a,b]$, it must be differentiable on the open interval (a,b). In addition, the derivative from the right at a must exist, and the derivative from the left at b must exist.

Ways in which a function can fail to be differentiable:

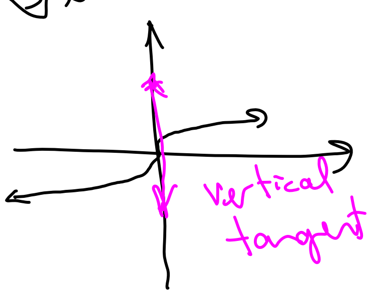
1. Sharp corner
2. Cusp
3. Vertical tangent
4. Discontinuity

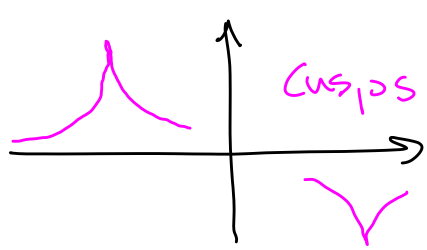
Cusp: occurs when $\lim\limits_{x\to a^-} f'(x) = +\infty$ and $\lim\limits_{x\to a^+} f'(x) = -\infty$

OR

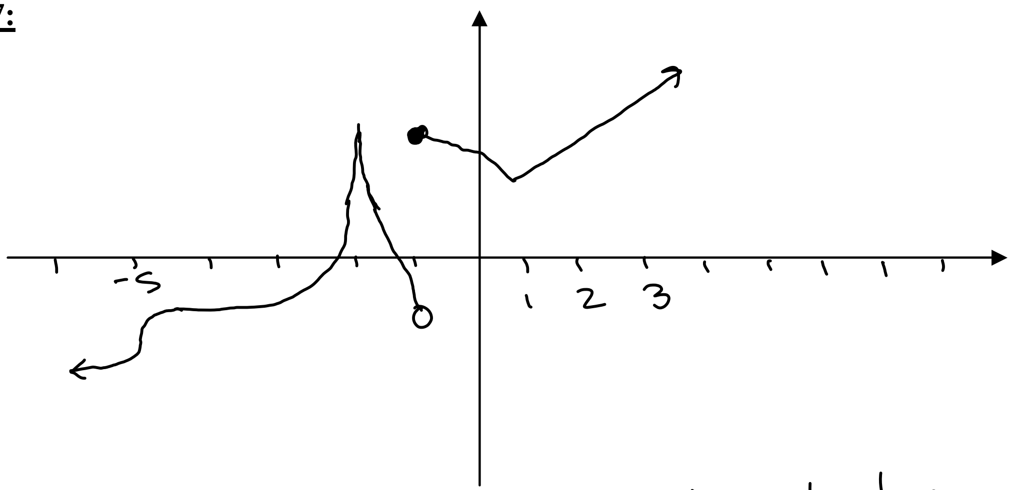
$\lim\limits_{x\to a^-} f'(x) = -\infty$ and $\lim\limits_{x\to a^+} f'(x) = +\infty$

$y = \sqrt[3]{x}$





Example 7:



This function is not differentiable at

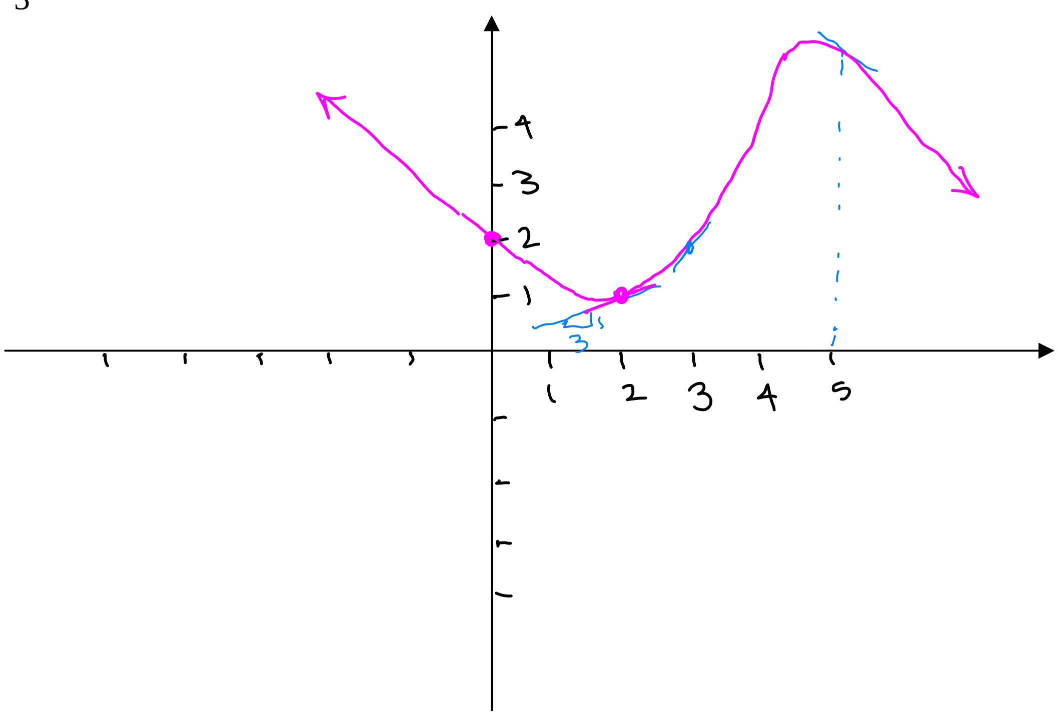
$x = -1$ (discontinuity)

$x = 1$ (sharp corner

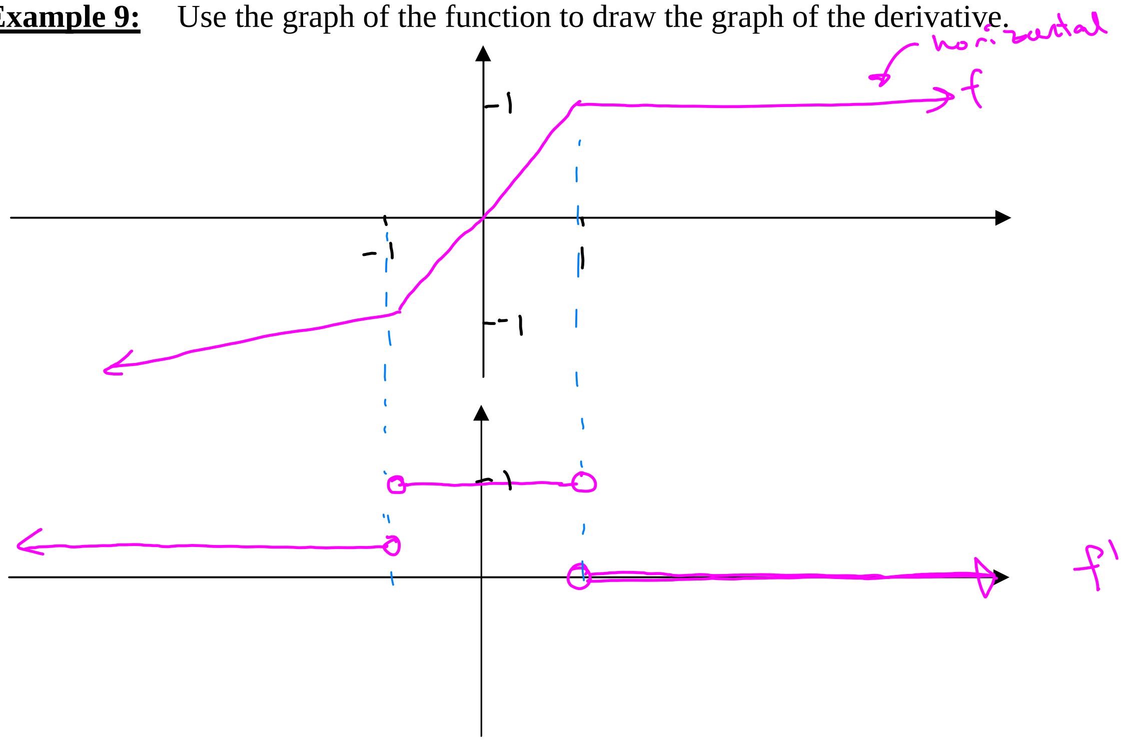
$x = -2$ (cusp)

$x = -5$ (vertical tangent)

Example 8: Sketch the graph of a function for which $f(0) = 2$, $f'(0) = -1$, $f(2) = 1$, $f'(2) = \frac{1}{3}$, $f'(3) > f'(2)$, and $f'(5) < 0$.



Example 9: Use the graph of the function to draw the graph of the derivative.



Example 10: Use the graph of the function to draw the graph of the derivative.