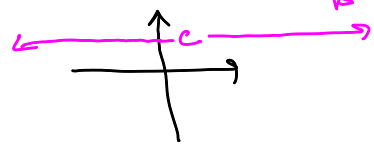
2.2: Basic Differentiation Rules and Rates of Change



$f'(x) = 0$ (slope is 0 tangent is horizontal)

Basic differentiation formulas:

1. $\frac{d}{dx}(c) = 0$ for any constant c.

   in other words, if $f(x) = c$, then $f'(x) = 0$

2. $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n. ← Power Rule

   if $f(x) = x^n$, then $f'(x) = nx^{n-1}$

3. $\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$

4. $\frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$

5. $\frac{d}{dx}\left[f(x)-g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right]$

Example 1: Find the derivative of $f(x) = 7$.

$f'(x) = 0$

Ex $1\frac{1}{2}$: $f(x) = 6x^4$

$f'(x) = 6\frac{d}{dx}(x^4) = 6(4x^{4-1}) = 6(4x^3)$

$= \boxed{24x^3}$

Example 2: Find the derivative of $f(x) = 5x^3 - x^7 + 12x$.

$f'(x) = 15x^2 - 7x^6 + 12x^0$

$= \boxed{15x^2 - 7x^6 + 12}$

Note:

$\frac{d}{dx}(kx) = k$

where k is a constant

Example 3: Find the derivative of $g(x) = x^{17} + x^{3/2}$.

$g(x) = x^{17} + x^{3/2}$

$g'(x) = 17x^{16} + \frac{3}{2}x^{\frac{3}{2}-1} = \boxed{17x^{16} + \frac{3}{2}x^{\frac{1}{2}}} = \boxed{17x^{16} + \frac{3\sqrt{x}}{2}}$

Recall:

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$
$$\frac{1}{x^n} = x^{-n}$$

Example 4: Find the derivative of $f(x) = \sqrt[5]{x} + \frac{1}{x^2}$.

Rewrite: $f(x) = x^{\frac{1}{5}} + x^{-2}$

$f'(x) = \frac{1}{5}x^{\frac{1}{5}-1} - 2x^{-2-1} = \frac{1}{5}x^{-4/5} - 2x^{-3}$

$= \frac{1}{5x^{4/5}} - \frac{2}{x^3} = \boxed{\frac{1}{5\sqrt[5]{x^4}} - \frac{2}{x^3}}$

Example 5: Find the derivative of $f(x) = \frac{2}{\sqrt[4]{x}}$.

$f(x) = 2x^{-\frac{1}{4}}$

$f'(x) = 2\left(-\frac{1}{4}\right)x^{-\frac{1}{4}-1} = -\frac{2}{4}x^{-5/4}$

$= \boxed{-\frac{1}{2\sqrt[4]{x^5}}}$

Example 6: Find the derivative of $h(x) = \left(\sqrt{x}\right)^5$.

$h(x) = \left(x^{1/2}\right)^5 = x^{5/2}$

$h'(x) = \frac{5}{2}x^{\frac{5}{2}-1} = \frac{5}{2}x^{3/2} = \boxed{\frac{5\sqrt{x^3}}{2}}$

Example 7: Find the derivative of $f(x) = -\sqrt[3]{6x^4}$.

$f(x) = -\sqrt[3]{6x^4} = -\left(6x^4\right)^{1/3} = -6^{1/3}x^{4/3}$

$f'(x) = -6^{1/3} \cdot \frac{4}{3}x^{\frac{4}{3}-1} = -6^{1/3} \cdot \frac{4}{3}x^{1/3} = -\frac{4}{3}(6x)^{1/3}$

OR $f(x) = -\sqrt[3]{6}\sqrt[3]{x^4} = -\sqrt[3]{6}\,x^{4/3}$

$f'(x) = -\sqrt[3]{6}\left(\frac{4}{3}x^{1/3}\right) = -\sqrt[3]{6}\left(\frac{4}{3}\right)\sqrt[3]{6} = -\frac{4}{3}\sqrt[3]{6x}$

$= \boxed{-\frac{4}{3}\sqrt[3]{6x}}$

Example 8: Find the derivative of $f(x) = \frac{10}{x^4}$.

Example 9: Find the derivative of $g(x)=\dfrac{2\sqrt{x}}{7}$.

Example 10: Find the derivative of $f(t)=\dfrac{3}{4t^2}-\sqrt[3]{7t}$.

Example 11: Find the derivative of $f(u)=\dfrac{7u^5+u^2-9\sqrt{u}}{u^2}$.

$$f(u)=\frac{7u^5}{u^2}+\frac{u^2}{u^2}-\frac{9u^{1/2}}{u^2}=7u^3+1-9u^{\frac{1}{2}-2}$$

$$=7u^3+1-9u^{-3/2}$$

$$f'(u)=21u^2+0-9\left(-\frac{3}{2}u^{-\frac{3}{2}-1}\right)$$

$$=\boxed{21u^2+\frac{27}{2}u^{-5/2}}=\boxed{21u^2+\frac{27}{2\sqrt{u^5}}}$$

Example 12: Find the equation of the tangent line to the graph of $f(x)=3x-x^2$ at the point $(-2,-10)$.

$$f'(x)=3-2x$$

Find slope: $m=f'(-2)=3-2(-2)=3+4=7$

$$y-y_1=m(x-x_1)$$

$$y-(-10)=7(x-(-2))$$

$$y+10=7(x+2)$$

$$y+10=7x+14$$

$$\boxed{y=7x+4}$$

Example 13: Find the point(s) on the graph of $f(x) = x^2 + 6x$ where the tangent line is horizontal.

Find derivative:

$f'(x) = 2x + 6$

For a horizontal tangent, the slope is 0. So set $f'(x) = 0$:

$2x + 6 = 0$

$2x = -6$

$x = -3$ It asked for a point, so we need the y-value:

$f(-3) = (-3)^2 + 6(-3)$
$= 9 - 18$
$= -9$

$\boxed{\text{Tangent is horizontal at } (-3, -9).}$

Definition: The *normal line* to a curve at the point *P* is defined to be the line passing through *P* that is perpendicular to the tangent line at that point.

Example 14: Determine the equation of the normal line to the curve $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$.

Recall: The slopes of perpendicular lines are opposite reciprocals.

So, we first find slope of tangent line:

$y = \frac{1}{x} = x^{-1}$

slope of tangent line is: $\frac{dy}{dx} = -1x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$

slope of tangent line at $(3, \frac{1}{3})$ is $-\frac{1}{3^2} = -\frac{1}{9} = m_1$

Slope of normal line is $m_2 = +\frac{9}{1} = 9$

Find eqn: $y - y_1 = m(x - x_1) \Longrightarrow y - \frac{1}{3} = 9(x - 3)$

$y - \frac{1}{3} = 9x - 27$

$y = 9x - 27 + \frac{1}{3}$

$y = 9x - \frac{81}{3} + \frac{1}{3}$

$\boxed{y = 9x - \frac{80}{3}}$

Derivatives of trigonometric functions:

Know these!

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

Example 15: Find the derivative of $y = 2\cos x - 4\tan x$.

$$\frac{dy}{dx} = 2(-\sin x) - 4(\sec^2 x) = \boxed{-2\sin x - 4\sec^2 x}$$

Example 16: Find the derivative of $y = \frac{\sin x}{4} + 3x^4 + \pi^2$.

$$y = \frac{1}{4}\sin x + 3x^4 + \pi^2$$

$$\frac{dy}{dx} = \frac{1}{4}\cos x + 12x^3 + 0 = \boxed{\frac{1}{4}\cos x + 12x^3}$$

Example 17: Determine the equation of the tangent line to the graph of $y = \sec x$ at the point where $x = \frac{\pi}{4}$.

Find the derivative: $\frac{dy}{dx} = \sec x \tan x$

Slope at $x = \frac{\pi}{4}$ is: $m = \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} \cdot \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} \cdot \frac{\sqrt{2}/2}{\sqrt{2}/2} = \frac{2}{\sqrt{2}} \cdot (1) = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Find y-value: $y\Big|_{x=\frac{\pi}{4}} = \sec\frac{\pi}{4} = \sqrt{2}$

$$y - y_1 = m(x - x_1)$$
$$y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$$
$$\boxed{y = \sqrt{2}\,x - \frac{\pi\sqrt{2}}{4} + \sqrt{2}}$$

Example 18: Find the points on the curve $y = \tan x - 2x$ where the tangent line is horizontal.

The derivative as a rate of change:

The <u>average rate of change</u> of $y = f(x)$ with respect to x over the interval $[x_0, x_1]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}, \text{ where } h = x_1 - x_0 \neq 0.$$

This is the same as the slope of the secant line joining points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$.

The <u>instantaneous rate of change</u> (or, equivalently, just the <u>rate of change</u>) of f when $x = a$ is the slope of the tangent line to graph of f at the point $(a, f(a))$.

Therefore, the instantaneous rate of change is given by the <u>derivative</u> f'.

Δ = delta = "change in"

<u>Example 19:</u> Find the average rate of change in volume of a sphere with respect to its radius r as r changes from 3 to 4. Find the instantaneous rate of change when the radius is 3.

Volume of a sphere: $V = \frac{4}{3}\pi r^3$ (r = radius)

Average rate of change in volume $= \frac{\Delta V}{\Delta r} = \frac{\text{change in } V}{\text{change in } r} = \frac{V_2 - V_1}{r_2 - r_1} = \frac{V(4) - V(3)}{4 - 3}$

$= \frac{\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(3)^3}{} = \frac{4}{3}\pi(64 - 27) = \frac{4}{3}\pi(37) = \boxed{\frac{148}{3}\pi}$ avg rate of change

Instantaneous rate of change: $V'(r) = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{\cancel{3}}\pi(\cancel{3}r^2) = 4\pi r^2$

At $r = 3$, this is $V'(3) = 4\pi(3)^2 = \boxed{36\pi}$

<u>Example 20:</u> Find the rate of change of the area of a circle with respect to (a) the diameter; (b) the circumference.

(a)

Area $= \pi r^2$

$A(r) = \pi r^2$

Diameter: $d = 2r$

$r = \frac{d}{2}$

Substitute $r = \frac{d}{2}$ into $A(r) = \pi r^2$:

$A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$

Now it's a function of d: $A(d) = \frac{\pi d^2}{4} = \frac{1}{4}\pi d^2$

$A'(d) = \frac{1}{4}\pi(2d) = \boxed{\frac{1}{2}\pi d}$

(b) Circumference: $C = 2\pi r = \pi d$

$C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$

Substitute into $A = \pi r^2$:

$A(C) = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{\pi C^2}{4\pi^2}$

$= \frac{C^2}{4\pi} = \frac{1}{4\pi}C^2$

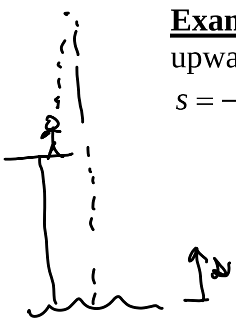
$A'(C) = \frac{1}{4\pi}(2C) = \boxed{\frac{C}{2\pi}}$

Velocity:

If the independent variable represents *time*, then the derivative can be used to analyze motion.

If the function $s(t)$ represents the position of an object, then the derivative $s'(t) = \frac{ds}{dt}$ is the <u>velocity</u> of the object. (velocity = instantaneous velocity)

(The velocity is the instantaneous rate of change in distance. The average velocity is the average rate of change in distance.)



<u>Example 21:</u> A person stands on a bridge 40 feet above a river. He throws a ball vertically upward with an initial velocity of 50 ft/sec. Its height (in feet) above the river after t seconds is $s = -16t^2 + 50t + 40$.

a) What is the velocity after 3 seconds?
b) How high will it go?
c) How long will it take to reach a velocity of 20 ft/sec?
d) When will it hit the water? How fast will it be going when it gets there?

a velocity: $v(t) = \frac{ds}{dt} = s'(t) = -32t + 50$

After 3 seconds: $s'(3) = -32(3) + 50 = -96 + 50 = \boxed{-46 \text{ ft/sec}}$

(it is going down)

b At maximum height, the velocity is 0 (velocity is positive going up, and negative going down)

Set $s'(t) = 0$: $-32t + 50 = 0$

$$50 = 32t$$

$$t = \frac{50}{32} = \frac{25}{16} \text{ seconds} = 1\tfrac{9}{16} \text{ sec}$$

Put $t = \frac{25}{16}$ into position function $s(t) = -16t^2 + 50t + 40$

$$s\left(\frac{25}{16}\right) = -16\left(\frac{25}{16}\right)^2 + 50\left(\frac{25}{16}\right) + 40 = \boxed{79.0625 \text{ ft}}$$

c set $s'(t) = 20$: $-32t + 50 = 20$

$$-32t = -30$$

$$t = \frac{-30}{-32} = \boxed{\frac{15}{16} \text{ sec}}$$

d when it hits the water, $s = 0$: $-16t^2 + 50t + 40 = 0$

Need quadratic formula - see next page

$$t = \frac{-50 \pm \sqrt{50^2 - 2(-16)(40)}}{2(-16)} \Rightarrow t \approx 3.785 \text{ sec}, \quad t \approx -0.66 \text{ sec}$$

Find velocity when it hits water: $s'(3.785) = -32(3.785)+50) = -71.12$ ft/sec

It is going at a speeed of 71.12 ft/sec when it hits water.

Example 22: Suppose a bullet is shot straight up at an initial velocity of 73 feet per second. If air resistance is neglected, its height from the ground (in feet) after t seconds is given by $h(t) = -16.1t^2 + 73t$.

a. The velocity after 2 seconds.
b. How high will the bullet go?
c. When will the bullet reach the ground?
d. How fast will it be traveling when it hits the ground?

$h'(t) = -32.2t + 73$

a) After 2 sec, velocity is $h'(2) = -32.2(2) + 73 = \boxed{8.6 \text{ ft/sec}}$

b) Set $h'(t) = 0$: (velocity is 0 at max height)

$$-32.2t + 73 = 0$$
$$73 = 32.2t$$
$$t \approx 2.267 \text{ sec}$$

$$h(2.267) = -16.1(2.267)^2 + 73(2.267) \approx \boxed{82.75 \text{ ft}}$$

c) when it hits ground, h is 0:

set $h(t) = 0$: $-16.1t^2 + 73t = 0$

$$t(-16.1t + 73) = 0$$
$$t = 0, \quad t = \frac{73}{16.1} \approx \boxed{4.53 \text{ sec}}$$

d) $h'(4.53)$
$= -32.2(4.53) + 73$
$= -72.866$

Speed is 72.866 ft/sec

Example 23: Suppose the position of a particle is given by $f(t) = t^4 - 32t + 7$. What is the velocity after 3 seconds? When is the particle at rest?

At rest: velocity $= 0$.

$$f'(t) = 4t^3 - 32$$
$$f'(t) = 4(3)^3 - 32 = 4(27) - 32 = \boxed{76}$$ velocity after 3 sec

Velocity is 0: $4t^3 - 32 = 0$

$$4t^3 = 32$$
$$t^3 = 8$$
$$t = 2$$

velocity $= 0$ at $t = 2$.